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POSITIVE RADIAL SOLUTIONS OF $\Delta u + \lambda f(u) = 0$ ON ANNULUS

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1. Introdution

We consider the behavior of positive radial solutions (or, briefly, p.r.s.) of the equation

$$\begin{aligned} \Delta u + \lambda f(u) &= 0 \qquad \text{in}\,\Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega = \{\, x \in \mathbf{R}^n \mid A < |x| < B \,\}$ is an annulus in $\mathbf{R}^n$, $n \geq 2$, $\lambda > 0$ and $f \geq 0$ is superlinear in u and satisfies $f(0) = 0$. The existence of solutions for this problem in a bounded domain has been proved under various sets of assumptions, always including a restriction on the growth of f at infinity[1],[3],[5]. Such a growth condition is, in general, necessary for star-shaped domains [11]. Since the annulus is not a star-shaped domain, there are no "natural" constraints for the growth of f. When Ω is a ball, all positive solutions of (1.1) have to be radially symmetric for any Lipschitz continuous $f(u)$ [7]. On the annulus, there are nonradial positive solutions of (1.1) for some nonlinearities [2],[9]. Here we limit ourselves to the radial solutions of (1.1). Recently, it is proved that the uniqueness of the positive solutions of

$$\begin{aligned} \Delta u + \lambda u + u^p &= 0 \qquad \text{in } B_R, \\ u &= 0 \qquad \text{on } \partial B_R, \end{aligned} \tag{1.2}$$

where $p \in (1, \frac{n+2}{n-2}]$ for $n \geq 3$, $p > 1$ for $n = 2$ and B_R is a ball of radius R in $\mathbf{R}^n$ [8],[12]. In the remaining case of a supercritical

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$p > \frac{n+2}{n-2}$, uniqueness is no longer valid [4]. These subtle phenomena also take place for the annulus [8]. Note that Pohozaev's identity is powerful to obtain uniqueness results of equation (1.2) for $1 < p \leq \frac{n+2}{n-2}$ in a ball or an annulus. When there is no growth condition imposed on f, uniqueness has been proved in a "thin domain" [10]. Assuming existence, we shall describe the behavior of upper bounds of $B - A$ as $A \to 0$ or $A \to \infty$. With these observations, we establish uniqueness of p.r.s. of problem (1.1) under an "inner radius" condition even if $f(u) = \lambda u + u^p$, $p > \frac{n+2}{n-2}$.

We assume that f satisfies the following conditions:

$$\text{(H-1) } f(0) = 0, \quad \lim_{u \to 0+} \frac{f(u)}{u} = 1,$$

$$\text{(H-2) } uf'(u) > f(u) > 0 \quad \text{in } u > 0,$$

$$\text{(H-3) } f(u) = O(u^p) \quad \text{as } u \to \infty \text{ with } p \geq 1.$$

Under the hypotheses (H-1) and (H-2), if (1.1) has a positive solution, then $\lambda \in (0, \lambda_1)$, where λ_1 is the first eigenvalue of the Laplacian $-\Delta$ with zero boundary data. In [9], it is proved that for any $\lambda \in (0, \lambda_1)$, there exists a positive radial solution of (1.1) provided f satisfies (H-1), (H-2) and (H-3)$'$ $uf(u) \geq 2(1+\epsilon)\int_0^u f(t)dt$ for u large and $\epsilon > 0$. We can give a simple proof different from that in [9].

Our techniques are a shooting method and Sturm's comparison principle.

2. Upper bounds of B depending on A

Since we are interested in radial solutions, we write (1.1) as

$$\begin{aligned} &u''(r) + \frac{n-1}{r}u'(r) + f(u(r)) = 0, \quad r \in (A, B), \\ &u(A) = u(B) = 0, \end{aligned} \tag{2.1}$$

when $n \geq 2$, $\lambda = 1$.

We assum that f satisfies (H-1) - (H-3). For p.r.s. of (2.1), a "phase-plane" analysis shows that there is a unique m such that $u'(m) = 0$

and so $u'(r) > 0$ for $A < r < m$ and $u'(r) < 0$ for $m < r < B$ [6]. We denote $h = u(m)$; the maximum of u.

Now, we define an "energy function" $H(r)$ by

$$H(r) = \frac{(u'(r))^2}{2} + F(u(r)),$$

where $F(u) = \int_0^u f(t)dt$. Then

$$H'(r) = -\frac{n-1}{r}(u'(r))^2 \leq 0. \tag{2.2}$$

First, we can give a universal upper bound of $B - A$. The main idea of estimate is included in [6].

PROPOSITION 2.1. *If $\lambda = 1$, u is a p.r.s. of (1 1), then*

$$B - A < \frac{3}{2}\pi + n - 1.$$

Proof. For $r \in [A, m]$, integrating H' from r to m gives

$$u'(r) \geq [2F(h) - 2F(u(r))]^{1/2}.$$

(H-1) and (H-2) imply

$$\begin{aligned} m - A &\leq \int_A^m \frac{u'(r)dr}{[2F(h) - 2F(u(r))]^{1/2}} \\ &< \int_A^m \frac{u'(r)dr}{[h^2 - u^2]^{1/2}} = \frac{\pi}{2}. \end{aligned}$$

Next, we define θ by $\tan\theta = \frac{u'}{u}$. Then for $r \in [m, B)$, $-\frac{\pi}{2} < \theta \leq 0$. We differentiate to get

$$\begin{aligned} \theta' &= \frac{u''u - u'^2}{u^2 + u'^2} \\ &= \frac{-\frac{n-1}{r}uu' - f(u)u - u'^2}{u^2 + u'^2} < -1 - \frac{n-1}{2r}\sin 2\theta. \end{aligned}$$

If $B - m > \pi + n - 1$, then for $n - 1 < r - m < \pi + n - 1$, we have $\frac{r}{n-1} \geq \frac{r-m}{n-1} \geq -\sin 2\theta$ and so $\theta' < -\frac{1}{2}$. Thus $\int_{m+n-1}^{m+\pi+n-1} \theta' dr < -\frac{\pi}{2}$. This contradicts $-\frac{\pi}{2} < \theta \leq 0$. Therefore $B - m \leq \pi + n - 1$ and $B - A < \frac{3}{2}\pi + n - 1$. □

Before estimating upper bound of $B - A$ sharply, we state the following theorem which can be found in [6].

THEOREM 2.2. *Let f be a continuous real function satisfying $f(u) > 0$ for $u > 0$ and $0 < \lambda < \infty$. Assume (H-1) and (H-3).*

(1) *If $p = 1$, there are constants $0 < d_1 \le d_2$ such that there exists a p.r.s. to problem (1.1) if $d_1 < B - A < d_2$ and no p.r.s. if $B - A < d_1$ or $B - A > d_2$.*
(2) *If $p > 1$, there is a constant $d > 0$ such that there is a p.r.s. to problem (1.1) if $0 < B - A < d$ and no p.r.s. if $B - A > d$.*

Let us fix $A > 0$ and denote $C_\lambda(A)$ the supremum of $B > A$ such that there exists a p.r.s. of (1.1). For simplicity, we assume $\lambda = 1$. Then, from (2.2), we have

$$u'(B)^2 < 2F(h) < u'(A)^2.$$

Since $h \to 0$ as $u'(A) \to 0$, a p.r.s. of the following equation with $u(A) = 0$ must be considered.

$$u''(r) + \frac{n-1}{r}u'(r) + u(r) = 0 \quad \text{on} \Omega. \tag{2.3}$$

Sturm's comparison theorem implies that the second zeros of positive solutions of (2.1) converges to the second zero of a solution of (2.3) as $u'(A) \to 0$. The linear equation (2.3) is a variant of Bessel's equation. We set

$$u(r) = r^{\frac{2-n}{2}} v(r).$$

Then the equation in (2.3) is transformed into the following Bessel's equation

$$v'' + \frac{v'}{r} + \left(1 - \frac{(\frac{n-2}{2})^2}{r^2}\right)v = 0. \tag{2.4}$$

This equation has a solution :

$$v(r) = Y_{\frac{n-2}{2}}(A)J_{\frac{n-2}{2}}(r) - J_{\frac{n-2}{2}}(A)Y_{\frac{n-2}{2}}(r),$$

where J_α and Y_α are the Bessel functions of the first and the second kind of order α respectively. The variants of constants $w(r) = r^{\frac{1}{2}}v(r)$ transforms (2.4) into

$$w'' + \left(1 - \frac{(\frac{n-2}{2})^2 - \frac{1}{4}}{r^2}\right)w = 0. \tag{2.5}$$

Then the equation (2.5) is decided into three types according to the dimension of space. Proposition 2.1. implies a well-known fact that $C_1(A) \to \pi$ as $A \uparrow \infty$. Shifting solutions of (2.5), we observe that $C_1(A)$ is monotone. Since $Y_{\frac{n-2}{2}}(r)$ has a singularity at 0, $C_1(A)$ converges to the first zero of $J_{\frac{n-2}{2}}(r)$ except 0 as $A \to 0$. We state these phenomena as follows.

THEOREM 2.3. *Let $C_\lambda(A)$ the supremum of $B > A$ such that there exists a p.r.s. of (1.1).*

(1) i) *If $n = 2$, then $C_1 < \pi$ and $C_1 \downarrow z(2)$ as $A \downarrow 0$, $C_1 \uparrow \pi$ as $A \uparrow$*

ii) *If $n = 3$, then $C_1 = \pi$.*

iii) *If $n \geq 4$, then $C_1 > \pi$ and $C_1 \uparrow z(n)$ as $A \downarrow 0$, $C_1 \downarrow \pi$ as $A \uparrow \infty$, where $z(n)$ is the first zero of $J_{\frac{n-2}{2}}(r)$ except 0.*

(2) $C_\lambda(A) = \frac{1}{\sqrt{\lambda}} C_1(\sqrt{\lambda} A)$.

Now, we recall uniqueness results in [10].

THEOREM 2.4. *If $B/A \leq (n-1)^{\frac{1}{n-2}}$ for $n \geq 3$, and $B/A \leq e$ for $n = 2$, then* (1.1) *can have at most one positive radial solution provided* (H-2).

COROLLARY 2.5. *For sufficiently large A depending only n, λ,* (1.1) *has at most one positive radial solution provided f satisfies* (H-1)-(H-3).

Proof. Assuming existence, the ratio B/A converges to 1 as $A \to \infty$, namely Ω become a thin domain. □

COROLLARY 2.6. *For any $\lambda \in (0, \lambda_1)$, there exists a positive radial solution of* (1.1) *provided that f satisfies* (H-1) - (H-3).

Proof. If $\lambda \in (0, \lambda_1)$, then $C_\lambda > C_{\lambda_1}$ since eigenfunctions with the first eigenvalue λ_1 are radially symmetric. Then existence results are obtained by Theorem 2.2. □

REMARK. The first Dirichlet eigenvalue λ_1 is characterized by

$$\lambda_1 = \left(\frac{C_1(A)}{B-A}\right)^2.$$

Of course, if $n = 3$,

$$\lambda_1 = (\frac{\pi}{B-A})^2,$$

where $\Omega = \{ x \in \mathbf{R}^n \mid A < |x| < B \}$ is an annulus in $\mathbf{R}^n$.

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