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# SOME ANALYTIC IRREDUCIBLE PLANE CURVE SINGULARITIES

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## 1. Introduction

Let $V = \{(z,y) : f(z,y) = z^n + Ay^\alpha z^p + y^\beta z^q + y^k = 0\}$ and $W = \{(z,y) : g(z,y) = z^n + By^\gamma z^s + y^\delta z^t + y^k = 0\}$ be germs of analytic irreducible subvarieties of a polydisc near the origin in $\mathbb{C}^2$ with $n < k$ and $(n,k) = 1$ where $A$ and $B$ are complex numbers. Assume that $V$ and $W$ are topologically equivalent near the origin. Then we denote this relation by $f \sim g$ for brevity. If $V$ and $W$ are analytically equivalent at the origin, then we write $f \approx g$. Otherwise, we write $f \not\approx g$. Note that $f \sim z^n + y^k$ and that $\frac{\alpha}{n-p} > \frac{k}{n}$ and $\frac{\beta}{n-q} > \frac{k}{n}$ if $n > p > q \geq 1$. If $f(z,y) = z^n + Ay^\alpha z^p + Cy^\beta z^q + y^k$ where $C$ is a nonzero number, then we may assume without loss of generality that $C = 1$ in $f(z,y)$ considering $f(\varepsilon^k z, \varepsilon^n y)$ for some number $\varepsilon$.

Then we are going to prove the following:

THEOREM 3.7. *Let $V = \{(z,y) : f = z^n + Ay^\alpha z^p + y^\beta z^q + y^k = 0\}$ and $W = \{(z,y) : g = z^n + By^\gamma z^s + y^\delta z^t + y^k = 0\}$ where $A$ and $B$ are complex numbers. Let $f \sim g \sim z^n + y^k$ at the origin with $(n,k) = 1$ and $n < k$. Assume that $1 \leq q < p \leq n-2$, $1 \leq \alpha < \beta \leq k-2$, $\alpha + p < \beta + q$, $\frac{\alpha}{n-p} > \frac{\beta}{n-q}$; $1 \leq t < s \leq n-2$, $1 \leq \gamma < \delta \leq k-2$, $\gamma + s < \delta + t$, $\frac{\gamma}{n-s} > \frac{\delta}{n-t}$. Then $f \approx g$ if and only if $(\alpha, p) = (\gamma, s)$, $(\beta, q) = (\delta, t)$ and $a^n = d^k = a^q d^\beta$, $Aa^p d^\alpha = a^n B$ for some nonzero numbers $a, d$. In detail, $f \approx g$ implies that $a^{n\beta + kq - nk} = d^{n\beta+kq-nk} = 1$ and $A^{n\beta+kq-nk} = B^{n\beta+kq-nk}$.*

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THEOREM 3.9. *Let* $f = z^n + A_1 y^{\alpha_1} z^{p_1} + \cdots + A_{t-1} y^{\alpha_{t-1}} z^{p_{t-1}} + y^{\alpha_t} z^{p_t} + y^k$ *where* $n < k$, $(n,k) = 1$, $n-2 \geq p_1 > \cdots > p_t \geq 1$, $\alpha_t \leq k-2$, $\alpha_1 + p_1 < \cdots < \alpha_t + p_t$, $\frac{\alpha_1}{n-p_1} > \cdots > \frac{\alpha_t}{n-p_t} > \frac{k}{n}$ *and each* $A_i = A_i(z,y)$ *is a unit in* ${}_2\mathcal{O}$ *for* $i = 1, \ldots, t-1$. *Let* $g = z^n + B_1 y^{\beta_1} z^{q_1} + \cdots + B_{s-1} y^{\beta_{s-1}} z^{q_{s-1}} + y^{\beta_s} z^{q_s} + y^k$ *where* $n-2 \geq q_1 > \cdots > q_s \geq 1$, $\beta_s \leq k-2$, $\beta_1 + q_1 < \cdots < \beta_s + q_s$, $\frac{\beta_1}{n-q_1} > \cdots > \frac{\beta_s}{n-q_s} > \frac{k}{n}$ *and each* $B_j = B_j(z,y)$ *is a unit in* ${}_2\mathcal{O}$ *for* $j = 1, \ldots, s-1$. *If* $f \approx g$ *then* $t = s$, $(\alpha_i, p_i) = (\beta_i, q_i)$ *for* $i = 1, \ldots, t$ *and* $A_i(0,0)^{n\alpha_t + kp_t - nk} = B_i(0,0)^{n\alpha_t + kp_t - nk}$ *for* $i = 1, \ldots, t-1$. *In particular, if the* $A_i$ *and* $B_j$ *are nonzero complex numbers and* $n\alpha_t + kp_t - nk = 1$ *with the same assumption above, then* $f \approx g$ *if and only if* $(\alpha_i, p_i) = (\beta_i, q_i)$ *for* $i = 1, \ldots, t = s$ *and* $A_i = B_i$.

REMARK 3.10. In Theorem 3.7 and Theorem 3.9 we can prove the same result with the following numerical assumption, $k \leq \alpha_1 + p_1 \leq \cdots \leq \alpha_t + p_t$ instead of $\alpha_1 + p_1 < \cdots < \alpha_t + p_t$. If not, the same result may not hold by the example below:

$$z^4 + Ay^6 z^2 + y^7 z + y^9 \approx z^4 + y^7 z + y^9.$$

for any number $A$.

Now we can apply the above fact to some examples as below. Consider the family of analytic irreducible plane curve singularities $f_c$ at the origin parametrically defined by $y = t^4$ and $z = t^9 + t^{10} + ct^{11}$ where $c$ is a number. Then for any $c$ $f_c \sim z^4 + y^9$ at the origin, but for any two numbers $c \neq d$ $f_c$ and $f_d$ can be proved analytically different at the origin [2]. Here is another proof. Write $f_c$ in terms of a Weierstrass polynomial as follows:

$$z^4 - 2(1+2c)y^5 z^2 - 4(1+c^2 y)y^7 z - ((1+c^2 y)^2 - y(1-2c)^2)y^9 = 0$$

Then by Theorem 3.9 we can prove that $f_c \approx f_d$ if and only if $c = d$ because this example satisfies the additional assumption $n\alpha_t + kp_t - nk = 1$ as in Theorem 3.9.

## 2. Known preliminaries

DEFINITION 2.1. Let $V = \{z \in \mathbb{C}^n : f(z) = 0\}$ and $W = \{z \in \mathbb{C}^n : g(z) = 0\}$ be germs of complex analytic hypersurfaces with isolated singular points at the origin. (i) $V$ and $W$ are said to be topologically equivalent at the origin if there is a germ at the origin of homeomorphisms $\phi : (U_1, 0) \to (U_2, 0)$ such that $\phi(V) = W$ and $\phi(0) = 0$ where $U_1$ and $U_2$ are open subsets containing the origin in $\mathbb{C}^n$. In this case denote this relation by $f \sim g$. (ii) $V$ and $W$ are said to be analytically equivalent at the origin if there is a germ at the origin of biholomorphisms $\psi : (U_1, 0) \to (U_2, 0)$ such that $\psi(V) = W$ and $\psi(0) = 0$ where $U_1$ and $U_2$ are open subsets containing the origin in $\mathbb{C}^n$. Then denote this relation by $f \approx g$. If not, we write $f \not\approx g$. Let ${}_n\mathcal{O}$ denote the ring of germs of holomorphic functions at the origin in $\mathbb{C}^n$.

THEOREM 2.2 [5]. *Let $f(z,y) = a_0 z^n + a_1 y^{\alpha_1} z^{n-1} + \cdots + a_n y^{\alpha_n}$ be irreducible in ${}_2\mathcal{O}$ where each $a_i$ is a unit in ${}_2\mathcal{O}$ if exists and the $\alpha_i$ are positive integers. Then $\frac{\alpha_i}{i} \geq \frac{\alpha_n}{n}$ for all $i$. Moreover, if $\alpha_n = nk$ for some integer $k$, then $\frac{\alpha_n}{n} = \frac{\alpha_i}{i}$ for all $i = 1, \ldots, n-1$.*

COROLLARY 2.3. *Let $f(z,y) = z^n + a_1 y^{\alpha_1} z^{n-1} + \cdots + a_{n-1} y^{\alpha_{n-1}} z + y^k$ with $(n,k) = 1$ where each $a_i$ is a unit in ${}_2\mathcal{O}$ if exists and the $\alpha_i$ are positive integers. Then $f$ is irreducible in ${}_2\mathcal{O}$ if and only if $\frac{k}{n} < \frac{\alpha_i}{i}$ for all $i \neq n$. Moreover, in this case $f \sim z^n + y^k$ in ${}_2\mathcal{O}$.*

THEOREM 2.4 (MATHER-YAU). *Assume that $V = \{f(z_1, \ldots, z_n) = 0\}$ and $W = \{g(z_1, \ldots, z_n) = 0\}$ have the isolated singular point at the origin. Then the following conditions are equivalent:*

(i) $f \approx g$.

(ii) *$A(f)$ is isomorphic to $A(g)$ as a $\mathbb{C}$-algebra where $A(f) = {}_n\mathcal{O}/(f, \Delta f)$, $A(g) = {}_n\mathcal{O}/(g, \Delta g)$ and $(f, \Delta f)$ is the ideal in ${}_n\mathcal{O}$ generated by $f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}$.*

(iii) *$B(f)$ is isomorphic to $B(g)$ as a $\mathbb{C}$-algebra where $B(f) = {}_n\mathcal{O}/(f, m\Delta f)$, $B(g) = {}_n\mathcal{O}/(g, m\Delta g)$ and $(f, m\Delta f)$ is the ideal in ${}_n\mathcal{O}$ generated by $f$ and $z_i \frac{\partial f_i}{\partial z_j}$ for all $i, j = 1, \ldots, n$.*

THEOREM 2.5 (ARNOLD [1]). *Assume that $n < k$, $(n,k) = 1$ and that $g \sim z^n + y^k$ where $g$ is holomorphic at the origin in $\mathbb{C}^2$. Then $g \approx z^n + y^k + \Sigma c_i P_i$ where each $c_i$ is a nonzero number if exists and $P_i = y^{\alpha_i} z^{\beta_i}$ with $1 \leq \beta_i \leq n-2$, $1 \leq \alpha_i \leq k-2$ and $n\alpha_i + k\beta_i > nk$.*

THEOREM 2.6 [6]. *Let* $f = z^n + y^k + uy^\alpha z^\beta$ *and* $g = z^n + y^k + vy^\gamma z^\delta$ *where* $n < k$, $(n,k) = 1$ *and* $u, v$ *are units in* ${}_2\mathcal{O}$, *and* $1 \leq \beta, \delta \leq n-2$; $1 \leq \alpha, \gamma \leq k-2$ *with* $n\alpha + k\beta > nk$ *and* $n\gamma + k\delta > nk$. *Then* $f \approx g$ *if and only if* $\alpha = \gamma$ *and* $\beta = \delta$.

THEOREM 2.7 [7]. *Let* $f = z^n + y^k + \Sigma c_i P_i$ *and* $g = z^n + y^k + \Sigma d_j Q_j$ *where* $n < k$, $(n,k) = 1$, *and each* $c_i$ *and* $d_j$ *are nonzero numbers if exist and* $P_i = y^{\alpha_i} z^{\beta_i}$, $Q_j = y^{\gamma_j} z^{\delta_j}$ *with* $1 \leq \alpha_i, \gamma_j \leq k-2$ *and* $1 \leq \beta_i$, $\delta_j \leq n-2$ *satisfying that* $n\alpha_i + k\beta_i > nk$ *and* $n\gamma_j + k\delta_j > nk$. *Let* $M(f) = Min\{\alpha_i + \beta_i : c_i \neq 0\}$, $M(g) = Min\{\gamma_j + \delta_j : d_j \neq 0\}$, $S(f) = \{(\alpha_i, \beta_i) : \alpha_i + \beta_i = M(f)\}$ *and* $S(g) = \{(\gamma_j, \delta_j) : \gamma_j + \delta_j = M(g)\}$. *Assume that* $f \approx g$. *Then we have the following:*

(i) *Let* $M(f) \geq k$. *Then* $S(f) = S(g)$ *as sets.*
(ii) *Let* $M(f) < k$ *and* $(\alpha, \beta) \in S(f)$ *such that* $\beta \leq \beta_i$ *for any* $(\alpha_i, \beta_i) \in S(f)$. *Then there is an element* $(\gamma, \delta) \in S(g)$ *such that* $\alpha = \gamma$, $\beta = \delta$ *and* $\delta \leq \delta_j$ *for any* $(\gamma_j, \delta_j) \in S(g)$.

## 3. Some analytic classification of irreducible plane curve singularities defined by $z^n + Ay^\alpha z^p + y^\beta z^q + y^k = 0$ with $(n,k) = 1$

LEMMA 3.1. *Let* $V = \{f = z^n + y^\alpha z^p + y^\beta z^q + y^k = 0\}$ *have an irreducible singularity at the origin in* $\mathbb{C}^2$ *where* $(n,k) = 1$, $n < k$, $1 \leq q < p \leq n-2$, $1 \leq \alpha < \beta \leq k-2$, $\alpha + p \leq \beta + q$. *Then we have the following:*

(i) *If* $m \geq \alpha + p - n + 1$, *then* $m > \frac{\alpha}{n-p} > \frac{k-\alpha}{p}$.
(ii) *If* $m \geq \beta + q - n + 1$, *then* $m > \frac{\beta}{n-q} > \frac{k-\beta}{q}$.
(iii) *If* $\frac{\alpha}{n-p} > \frac{\beta}{n-q}$, *then* $\frac{\beta}{n-q} > \frac{\beta-\alpha}{p-q}$.
(iv) *If* $\beta + q \geq k$, *then* $\frac{\beta-\alpha}{p-q} > \frac{k-\alpha}{p} > \frac{k-\beta}{q}$.
(v) *If* $\alpha + p \geq k$, *then* $\alpha + p - k + 1 \geq \frac{p}{k-\alpha}$.
(vi) *If* $\beta + q \geq k$, *then* $\beta + q - k + 1 \geq \frac{q}{k-\beta}$.

*Proof.* By Corollary 2.3, note that $f$ is irreducible in ${}_2\mathcal{O}$ if and only if $\frac{\alpha}{n-p} > \frac{k}{n}$ and $\frac{\beta}{n-q} > \frac{k}{n}$. Let us prove subcases as follows: It is trivial that $\alpha + p > n$ and $\beta + q > n$.

(i) Observe that $\alpha + p - n + 1 > \frac{\alpha}{n-p}$ if and only if $(n-p-1)(\alpha + p - n) > 0$ and that $\frac{\alpha}{n-p} > \frac{k}{n}$ if and only if $\frac{\alpha}{n-p} > \frac{k-\alpha}{p}$.

(ii) Use the similar method as in the case (i).

(iii) Just compute $\frac{\alpha}{\beta} - 1 > \frac{n-p}{n-q} - 1$. Then $\frac{\alpha-\beta}{\beta} > \frac{q-p}{n-q}$ implies $\frac{\beta}{n-q} > \frac{\beta-\alpha}{p-q}$.

(iv) To prove $\frac{k-\alpha}{p} > \frac{k-\beta}{q}$, note that $\frac{1}{k-\beta} > \frac{1}{k-\alpha}$ since $\beta > \alpha$. Then $\frac{\beta+q-k}{k-\beta} > \frac{\alpha+p-k}{k-\alpha}$ and so $\frac{q}{k-\beta} > \frac{p}{k-\alpha}$. Next, to prove $\frac{\beta-\alpha}{p-q} > \frac{k-\alpha}{p}$, note that $\frac{q}{k-\beta} > \frac{p}{k-\alpha}$ if and only if $\frac{q}{p} - 1 > \frac{k-\beta}{k-\alpha} - 1$ if and only if $\frac{\beta-\alpha}{p-q} > \frac{k-\alpha}{p}$.

(v) See that $\alpha+p-k+1 \geq \frac{p}{k-\alpha}$ if and only if $(k-p-1)(\alpha+p-k) \geq 0$.

(vi) Use the similar method as in the case (v).

DEFINITION 3.2. Let $\mathbb{N}$ be the set of positive intergers and define $\leq$ on $\mathbb{N} \times \mathbb{N} = \{(\alpha, p) : \alpha \in \mathbb{N},\ p \in \mathbb{N}\}$ with the following property:

(i) $(\alpha, p) = (\beta, q)$ if and only if $\alpha = \beta$ and $p = q$.

(ii) $(\alpha, p) \leq (\beta, q)$ if $\alpha \leq \beta$ and $p \leq q$. Also, if $(\alpha, p) \leq (\beta, q)$ and $\alpha + p < \beta + q$, then we write $(\alpha, p) < (\beta, q)$.

Now, before we get the desired result let us introduce some notations as follows. Assume that $f \sim g \sim z^n + y^k$ with $n < k$ and $(n, k) = 1$. Let $f = z^n + Ay^\alpha z^p + y^\beta z^q + y^k$ and $g = z^n + By^\gamma z^s + y^\delta z^t + y^k$ where $A$ and $B$ are complex numbers, $1 \leq q < p \leq n-2$, $1 \leq \alpha < \beta \leq k-2$, $1 \leq t < s \leq n-2$ and $1 \leq \gamma < \delta \leq k-2$. If $f \approx g$, then by definition, there is a biholomorphic mapping $\phi : (U_1, 0) \to (U_2, 0)$ such that $f \circ \phi = ug$ where $U_1$ and $U_2$ are open subsets containing the origin and $u$ is a unit in ${}_2\mathcal{O}$. Write $\phi(z, y) = (H, L)$ as below:

$$H = H(z, y) = az + by + H_2 + H_3 + \cdots \quad \text{and}$$
$$L = L(z, y) = cz + dy + L_2 + L_3 + \cdots$$

where $H_n$ and $L_n$ are homogeneous polynomials of degree $n$ with $H_n = a_{n,0}z^n + a_{n-1,1}z^{n-1}y + \cdots + a_{0,n}y^n$ and $L_n = b_{n,0}z^n + b_{n-1,1}z^{n-1}y + \cdots + b_{0,n}y^n$.

Note that $ad-bc \neq 0$. Then $f \circ \phi(z, y) = H^n + AL^\alpha H^p + L^\beta H^q + L^k = u(z^n + By^\gamma z^s + y^\delta z^t + y^k)$ where $u$ is a unit in ${}_2\mathcal{O}$. We know that $b = 0$ because $f \sim g \sim z^n + y^k$ with $n < k$ and $(n, k) = 1$ implies that $\alpha + p$, $\beta + q$, $\gamma + s$ and $\delta + t$ are all greater than $n$.

DEFINITION 3.3. If the coefficient of monomial $y^l z^m$ must be zero in the expansion of $H^i L^j$ where $H^i L^j$ is one of $H^n$, $L^\alpha H^p$, $L^\beta H^q$ and $L^k$ in $f \circ \phi$ as we have seen just before Definition 3.3, then we write $y^l z^m \notin H^i L^j$ and otherwise, we write $y^l z^m \in H^i L^j$. Also if the homogeneous polynomial of degree $n$, $H_n$ in $H$ cannot be divisible analytically by $z$ then denote this relation by $z \nmid H_n$. Similary, if $L_n$ in $L$ cannot be divisible analytically by $y$, we write $y \nmid L_n$.

THEOREM 3.4. *Let* $V = \{(z,y) : f = z^n + y^\alpha z^p + y^\beta z^q + y^k = 0\}$ *and* $W = \{(z,y) : g = z^n + y^\gamma z^s + y^\delta z^t + y^k = 0\}$. *Let* $f \sim g \sim z^n + y^k$ *at the origin with* $(n,k) = 1$ *and* $n < k$. *Assume that* $n - 2 \geq p > q \geq 1$, $1 \leq \alpha < \beta \leq k-2$, $\beta + q \geq \alpha + p \geq k$, $\frac{\alpha}{n-p} > \frac{\beta}{n-q}$; $n - 2 \geq s > t \geq 1$, $1 \leq \gamma < \delta \leq k-2$, $\delta + t \geq \gamma + s \geq k$, $\frac{\gamma}{n-s} > \frac{\delta}{n-t}$. *If* $f \approx g$, *then* $(\alpha, p) = (\gamma, s)$ *and* $(\beta, q) = (\delta, t)$.

*Proof.* Suppose that $f \approx g$. Recall the definition of $f \approx g$ as we have seen just before Definition 3.3. First we are going to prove the following cases:

(I) (i) $y^\alpha z^p \notin H^n$, (ii) $y^\alpha z^p \notin L^k$ and (iii) $y^\alpha z^p \notin L^\beta H^q$.
(II) (i) $y^\beta z^q \notin H^n$, (ii) $y^\beta z^q \notin L^k$ and (iii) $y^\beta z^q \notin L^\alpha H^p$.
(III) (i) $y^k \notin H^n$, (ii) $y^k \notin L^\alpha H^p$ and (iii) $y^k \notin L^\beta H^q$.

Let $m$ be the smallest positive integer such that $z \nmid H_m$ in $H$ if exists and $r$, the smallest positive integer such that $y \nmid L_r$ in $L$ if exists. We prove each case as below. Inequality that $m \geq \alpha + p - n + 1$ will be proved inside the proof of the case (i) of (I).

(I)(i) $y^\alpha z^p \notin H^n$: If there is no such $m$, there is nothing to prove. If exists, it is enough to show that $p + (n-p)m > \alpha + p$, that is, $m > \frac{\alpha}{n-p}$. Note that $m \geq k - n + 1$. We are going to prove this case by following two steps.

($\mathrm{i}_a$) Let $n - 1 + m \geq \alpha + p$: By Lemma 3.1, it is trivial.

($\mathrm{i}_b$) Let $n-1+m < \alpha+p$: Consider the monomial $y^m z^{n-1}$. Note that $y^m z^{n-1} \in H^n$ and $m < \alpha < \beta$. Then $y^m z^{n-1}$ does not belong to $L^\alpha H^p$ and $L^\beta H^q$. Also $y^m z^{n-1} \notin ug$, because $m + n - 1 < \alpha + p = \gamma + s \leq \delta + t$ by Theorem 2.7. Since $f \approx g$, it remains to show that $y^m z^{n-1} \notin L^k$, which would imply a contradiction. Assume that $y^m z^{n-1} \in L^k$ and there exists such $r$. Then $m + r(k-m) \leq m + n - 1$, i.e., $r(k-m) \leq n - 1$. First we claim that

$$r(k-m) = n - 1 \quad \cdots (A)$$

If $r(k-m) < n-1$, consider $y^m z^{r(k-m)} \in L^k$. But $y^m z^{r(k-m)}$ does not belong to $H^n$, $H^p L^\alpha$, $H^q L^\beta$ and $ug$ because $m+r(k-m) < n-1+m < \alpha+p \leq \beta+q$ and $m < \gamma < \delta < k$. Thus we proved the first claim. Next, considering $y^{k-1}z^r \in L^k$, we claim that

$$m(n-r) = k-1 \quad \cdots (B)$$

Since $\alpha + p > n - 1 + m = r(k-m) + m \geq k - 1 + r$ by (A), $y^{k-1}z^r \notin L^\alpha H^p$ and $L^\beta H^q$. In order to prove that $y^{k-1}z^r \notin ug$, note that $k-1+r < \alpha+p = \gamma+s < \delta+t$. So $r < t < s$ and then $y^{k-1}z^r \notin ug$. So $y^{k-1}z^r$ would belong to $H^n$. Thus we get $k-1+r \geq r+(n-r)m$, that is, $k-1 \geq (n-r)m$. If $k-1 > (n-r)m$, then consider $z^r y^{(n-r)m} \in H^n$. But we see that $y^{(n-r)m}z^r$ does not belong to $L^k$, $L^\alpha H^p$, $L^\beta H^q$ and $ug$. Therefore we proved the claim (B). But, by (A) and (B), $\frac{k}{n} = \frac{1+m}{1+r}$. Since $(n,k) = 1$, $r < p$ and $m < \alpha$, it would be a contradiction. Thus we get the result $y^m z^{n-1} \notin L^k$, which is the desired contradiction. So $y^\alpha z^p \notin H^n$ and $n-1+m \geq \alpha + p$.

(ii) $y^\alpha z^p \notin L^k$: It is enough to show that $\alpha + p < \alpha + (k-\alpha)r$, i.e., $r > \frac{p}{k-\alpha}$ if there is the smallest positive integer $r$ such that $y \nmid L_r$, otherwise it is trivial. We prove this inequality by the following two steps.

(ii$_a$) Let $k-1+r > \alpha+p$: By Lemma 3.1, $r > \frac{p}{k-\alpha}$.

(ii$_b$) Let $k-1+r \leq \alpha+p$: Note that $r < q < p$. Consider $y^{k-1}z^r \in L^k$. Then $y^{k-1}z^r \notin L^\alpha H^p$, $L^\beta H^q$ and $ug$ because $k-1+r \leq \alpha+p = \gamma+s < \delta+t$ by Theorem 2.7. Since $f \approx g$, $y^{k-1}z^r$ would belong to $H^n$ and then we get an inequality $k-1+r \geq r+(n-r)m$ if there is the smallest integer $m$ such that $z \nmid H_m$, otherwise it is trivial. Claim that

$$k-1 = (n-r)m \quad \cdots (C)$$

If $k-1 > (n-r)m$, then $y^{(n-r)m}z^r \notin L^k$, $L^\alpha H^p$, $L^\beta H^q$ and $ug$, but $y^{(n-r)m}z^r \in H^n$. It would be a contradiction. Thus we proved the equality (C). Next consider $y^m z^{n-1} \in H^n$. Note that $m+n-1 < \alpha+p < \beta+q$ because of the following fact: $\alpha+p \geq k-1+r = (n-r)m+r > m+n-1$ by (C). So $y^m z^{n-1} \notin L^\alpha H^p$, $L^\beta H^q$ and $ug$, noting that $m+n-1 < \alpha+p = \gamma+s < \delta+t$ implies $m < \gamma < \delta$. Therefore $y^m z^{n-1} \in L^k$. Then we would get an inequality $m+n-1 \geq m+(k-m)r$, that is, $n-1 \geq (k-m)r$. If $n-1 > (k-m)r$, then consider $y^m z^{(k-m)r} \in L^k$. But $y^m z^{(k-m)r} \notin H^n$ $L^\alpha H^p$, $L^\beta H^q$ and $ug$. Therefore we can get

$$n - 1 = (k - m)r \ \cdots (D).$$

By (C) and (D), $\frac{k}{n} = \frac{1+m}{1+r}$. Since $(n,k) = 1$, $r < p$ and $m < \alpha$ it would be a contradiction. Thus we proved $y^{k-1}z^r \notin H^n$. Therefore $y^\alpha z^p \notin L^k$.

(iii) $y^\alpha z^p \notin L^\beta H^q$: If $\alpha + p < \beta + q$, it is trivial. If $\alpha + p = \beta + q$, recall the coefficient $b$ in $H = az + by + H_2 + \cdots$ and the coefficient $c$ in $L = cz + dy + L_2 + \cdots$. Note that $b = 0$. So it is enough to show that $c = 0$. If $c \neq 0$, then $y^{k-1}z \in L^k$ would not belong to $L^\alpha H^p$, $L^\beta H^q$ because $\alpha + p \geq k$ and $p > q \geq 2$, and $y^{k-1}z \notin ug$. So it remains to show that $y^{k-1}z \notin H^n$ because $f \approx g$. If $y^{k-1}z \in H^n$, then $k \geq 1 + m(n-1) \geq 1 + (k - n + 1)(n-1)$ and so $0 \geq (k-n)(n-2)$. Since $n > 2$, it is impossible.

(II)(i) $y^\beta z^q \notin H^n$: If there is the smallest integer $m$ such that $z \nmid H_m$, then it is enough to show that $q + (n-q)m > \beta + q$, i.e., $m > \frac{\beta}{n-q}$. Since $m \geq \alpha + p - n + 1$ by the case (i) of (I), it is clear by Lemma 3.1 and assumption.

(ii) $y^\beta z^q \notin L^\alpha H^p$: Suppose there is the smallest integer $m$ such that $z \nmid H_m$, otherwise it is trivial. Then it is enough to prove that $q + (p-q)m + \alpha > \beta + q$, i.e., $m > \frac{\beta-\alpha}{p-q}$. Since $m \geq \alpha + p - n + 1 > \frac{\alpha}{n-p} > \frac{\beta}{n-q} > \frac{\beta-\alpha}{p-q}$ by the case (i) of (I), Lemma 3.1 and assumption, it is trivial.

(iii) $y^\beta z^q \notin L^k$: Assume that there is the smallest integer $r$ such that $y \nmid L_r$, otherwise it is trivial. Then it is enough to prove that $\beta + (k-\beta)r > \beta + q$, that is, $r > \frac{q}{k-\beta}$. Consider the following two cases ($\text{iii}_a$) and ($\text{iii}_b$).

($\text{iii}_a$) $k - 1 + r > \beta + q$: It is trivial by Lemma 3.1.

($\text{iii}_b$) Let $k - 1 + r \leq \beta + q$: Note that $r < q < p$. Consider $y^{k-1}z^r \in L^k$. It is enough to show that $y^{k-1}z^r \in f \circ \phi$ but $y^{k-1}z^r \notin ug$. Since $f \approx g$ and $y^{k-1}z^r \notin L^\beta H^q$, it remains to show that $y^{k-1}z^r \notin H^n$, $L^\alpha H^p$ and $ug$ as follows.

($\text{iii}_{b_1}$) $y^{k-1}z^r \notin H^n$: It is enough to show that $r + (n-r)m > k - 1 + r$. It is proved by the fact that $k - 1 + r < \beta + q < r + (q-r)m + \beta < r + (n-r)m$ because $m > \frac{\beta}{n-q}$ by the case (i) of (I).

(iii$_{b_2}$) $y^{k-1}z^r \notin L^\alpha H^p$: It is enough to show that $r+(p-r)m+\alpha > k-1+r$. It is proved by the fact that $k-1+r < \beta+q < r+(q-r)m+\beta < r+(p-r)m+\alpha$ because $m > \frac{\beta-\alpha}{p-q}$ by the case (i) of (I).

(iii$_{b_3}$) $y^{k-1}z^r \notin ug$: Note that $y^{k-1}z^r \in f\circ\phi$ by (iii$_{b_1}$) and (iii$_{b_2}$). Since $f \approx g$, then $y^{k-1}z^r \in ug$. So $(k-1,r) > (\delta,t)$ because $r < p = s$. Since $y^\delta z^t \in ug$ and $f \approx g$, it is enough to show that $y^\delta z^t \notin f\circ\phi$. Since $\delta+t < k-1+r \leq \beta+q$ then $y^\delta z^t \notin L^k$ and $L^\beta H^q$. It remains to show that $y^\delta z^t \notin H^n$ and $L^\alpha H^p$. First, to prove that $y^\delta z^t \notin H^n$, it is enough to show that $t+(n-t)m > \delta+t$, that is, $m > \frac{\delta}{n-t}$. Since $\beta+q > \delta+t$ and $q > r \geq t$, then $\frac{\beta}{n-q} > \frac{\delta}{n-t}$. Thus it is proved. Next, to prove that $y^\delta z^t \notin L^\alpha H^p$, it is enough to show that $t+(p-t)m+\alpha > \delta+t$, i.e., $m > \frac{\delta-\alpha}{p-t}$. Since $\frac{\alpha}{n-p} > \frac{\delta}{n-t}$ and $p > t$, then $\frac{\alpha}{n-p} > \frac{\delta-\alpha}{p-t}$. Thus we proved that $y^\delta z^t \notin f\circ\phi$ and $y^{k-1}z^r \notin ug$.

(III)(i) $y^k \notin H^n$: It is enough to show that $mn > k$, which is trivial.

(ii) $y^k \notin L^\alpha H^p$: It suffices to prove that $\alpha+mp > k$, that is, $m > \frac{k-\alpha}{p}$. By Lemma 3.1, it is clear.

(iii) $y^k \notin L^\beta H^q$: As in the above case (ii) of (III), we can prove it, similarly.

Thus we proved the cases (I), (II) and (III). Therefore, $f\circ\phi = ug$ implies that $(\alpha,p) = (\gamma,s)$ and $(\beta,q) \geq (\delta,t)$ by Theorem 2.7.

Next, applying the same method to $g\circ\phi^{-1} = u^{-1}f$ then we get $(\delta,t) \geq (\beta,q)$. Thus it is proved.

THEOREM 3.5. *Let $V = \{(z,y) : f = z^n+y^\alpha z^p+y^\beta z^q+y^k = 0\}$ and $W = \{(z,y) : g = z^n+y^\gamma z^s+y^\delta z^t+y^k = 0\}$. Let $f \sim g \sim z^n+y^k$ at the origin with $(n,k)=1$ and $n<k$. Assume that $n-2 \geq p > q \geq 1$, $1 \leq \alpha < \beta \leq k-2$, $\alpha+p < k \leq \beta+q$ and $\frac{\alpha}{n-p} > \frac{\beta}{n-q}$; $n-2 \geq s > t \geq 1$, $1 \leq \gamma < \delta \leq k-2$, $\gamma+s < k \leq \delta+t$ and $\frac{\gamma}{n-s} > \frac{\delta}{n-t}$. If $f \approx g$, then $(\alpha,p) = (\gamma,s)$ and $(\beta,q) = (\delta,t)$.*

*Proof.* It is enough to prove the following cases by Theorem 2.7 as we have seen in the proof of Theorem 3.4:

(I)(i) $y^\alpha z^p \notin H^n$, (ii) $y^\alpha z^p \notin L^\beta H^q$ and (iii) $y^\alpha z^p \notin L^k$.

(II)(i) $y^\beta z^q \notin H^n$, (ii) $y^\beta z^q \notin L^\alpha H^p$ and (iii) $y^\beta z^q \notin L^k$.

(III)(i) $y^k \notin H^n$, (ii) $y^k \notin L^\alpha H^p$ and (iii) $y^k \notin L^\beta H^q$.

Let $m$ be the smallest integer such that $z \nmid H_m$ in $H$ if exists and $r$ the smallest integer such that $y \nmid L_r$ in $L$ if exists. Now we are going to prove the above case, respectively. Note that $m \geq \alpha + p - n + 1$ if exists.

(I)(i) $y^\alpha z^p \notin H^n$: It is enough to show that $p + (n-p)m > \alpha + p$, i.e., $m > \frac{\alpha}{n-p}$. By Lemma 3.1, it is clear.

(ii) $y^\alpha z^p \notin L^\beta H^q$: If $\alpha + p < \beta + q$, then it is clear.

(iii) $y^\alpha z^p \notin L^k$: If $\alpha + p < k$, then it is trivial.

(II)(i) $y^\beta z^q \notin H^n$: It suffices to prove that $q + (n-q)m > \beta + q$, that is, $m > \frac{\beta}{n-q}$. By Lemma 3.1, it is clear.

(ii) $y^\beta z^q \notin L^\alpha H^p$: It remains to show that $q + (p-q)m + \alpha > \beta + q$, that is, $m > \frac{\beta - \alpha}{p-q}$. By Lemma 3.1, it is clear.

(iii) $y^\beta z^q \notin L^k$: By the similar method as we have seen in the proof of (iii) in (II), Theorem 3.4, we can prove that $y^\beta z^q \notin L^k$.

(III)(i) $y^k \notin H^n$: We need to show that $mn > k$, which is trivial.

(ii) $y^k \notin L^\alpha H^p$: It is enough to show that $\alpha + mp > k$, that is, $m > \frac{k-\alpha}{p}$. By Lemma 3.1, it is clear.

(iii) $y^k \notin L^\beta H^q$: It is enough to show that $\beta + mq > k$, that is, $m > \frac{k-\beta}{q}$. By Lemma 3.1, it is trivial.

THEOREM 3.6. *Let* $V = \{(z,y) : f = z^n + y^\alpha z^p - y^\beta z^q + y^k = 0\}$ *and* $W = \{(z,y) : g = z^n + y^\gamma z^s + y^\delta z^t + y^k = 0\}$. *Let* $f \sim g \sim z^n + y^k$ *at the origin with* $(n,k) = 1$ *and* $n < k$. *Assume that* $n - 2 \geq p > q \geq 1$, $1 \leq \alpha < \beta \leq k-2$, $\alpha + p < \beta + q < k$, $\frac{\alpha}{n-p} > \frac{\beta}{n-q}$; $n - 2 \geq s > t \geq 1$, $1 \leq \gamma < \delta \leq k - 2$, $\gamma + s < \delta + t$ *and* $\frac{\gamma}{n-p} > \frac{\delta}{n-t}$. *If* $f \approx g$, *then* $(\alpha, p) = (\gamma, s)$ *and* $(\beta, q) = (\delta, t)$.

*Proof.* It is enough to prove the following cases, by Theorem 2.7 as we have seen in the proof of Theorem 3.4:

(I)(i) $y^\alpha z^p \notin H^n$, (ii) $y^\alpha z^p \notin L^\beta H^q$ and (iii) $y^\alpha z^p \notin L^k$.

(II)(i) $y^\beta z^q \notin H^n$, (ii) $y^\beta z^q \notin L^\alpha H^p$ and (iii) $y^\beta z^q \notin L^k$.

(III)(i) $y^k \notin H^n$, (ii) $y^k \notin L^\alpha H^p$ and (iii) $y^k \notin L^\beta H^q$.

Let $m$ be the smallest integer such that $z \nmid H_m$ in $H$ if exists and $r$, the smallest integer such that $y \nmid L_r$ in $L$, if exists. Now let us prove each case, respectively. Note that $m \geq \alpha + p - n + 1$ if exists and $(\alpha, p) = (\gamma, s)$ by Theorem 2.7.

(I)(i) $y^\alpha z^p \notin H^n$: If there is such $m$, it is enough to show that $p + (n-p)m > \alpha + p$, that is, $m > \frac{\alpha}{n-p}$. By Lemma 3.1, it is trivial.

(ii) $y^\alpha z^p \notin L^\beta H^q$: If $\alpha + p < \beta + q$, then it is clear.

(iii) $y^\alpha z^p \notin L^k$: If $\alpha + p < k$, then it is trivial.

(II)(i) $y^\beta z^q \notin H^n$: By the similar method as in the subcase (i) of (I) and Lemma 3.1, we can prove it.

(ii) $y^\beta z^q \notin L^\alpha H^p$: Since $p > q$ it is enough to prove that $q + (p - q)m + \alpha > \beta + q$, that is, $m > \frac{\beta-\alpha}{p-q}$. By Lemma 3.1, it is proved.

(iii) $y^\beta z^q \notin L^k$: If $\beta + q < k$, then it is trivial.

(III)(i) $y^k \notin H^n$: If there is such $m$, it is trivial to show that $mn > k$.

(ii) $y^k \notin L^\alpha H^p$: We need to show that $\alpha + mp > k$, i.e., $m > \frac{k-\alpha}{p}$. By Lemma 3.1, it is trivial.

(iii) $y^k \notin L^\beta H^q$: It is enough to show that $\beta + mq > k$. By Lemma 3.1, it is clear.

THEOREM 3.7. *Let* $V = \{(z,y) : f = z^n + Ay^\alpha z^p + y^\beta z^q + y^k = 0\}$ *and* $W = \{(z,y) : g = z^n + By^\gamma z^s + y^\delta z^t + y^k = 0\}$ *where* $A$ *and* $B$ *are complex numbers. Let* $f \sim g \sim z^n + y^k$ *at the origin with* $(n,k) = 1$ *and* $n < k$. *Assume that* $n - 2 \geq p > q \geq 1$, $1 \leq \alpha < \beta \leq k - 2$, $\beta + q > \alpha + p$, $\frac{\alpha}{n-p} > \frac{\beta}{n-q}$; $n - 2 \geq s > t \geq 1$, $1 \leq \gamma < \delta \leq k - 2$, $\delta + t > \gamma + s$, $\frac{\gamma}{n-s} > \frac{\delta}{n-t}$. *Then* $f \approx g$ *if and only if* $(\alpha,p) = (\gamma,s)$, $(\beta,q) = (\delta,t)$ *and* $a^n = d^k = a^q d^\beta$, $Aa^p d^\alpha = a^n B$ *for some nonzero numbers* $a, d$. *In detail,* $f \approx g$ *implies that* $a^{n\beta+kq-nk} = d^{n\beta+kq-nk} = 1$ *and* $A^{n\beta+kq-nk} = B^{n\beta+kq-nk}$.

*Proof.* Let $\phi : (U_1, 0) \to (U_2, 0)$ be a biholomorphic mapping such that $f \circ \phi = H^n + AL^\alpha H^p + L^\beta H^q + L^k = ug$ where $u = u_{00} + u_{10}z + u_{01}y + \cdots$ is a unit in ${}_2\mathcal{O}$ as we have seen in Definition 3.3, $H = az + by + H_1 + \cdots$ and $L = cz + dy + L_1 + \cdots$. Note that $b = 0$ and $(\alpha,p) = (\gamma,s)$ by Theorem 2.7. First, to prove $(\beta,q) = (\delta,t)$ it is enough to consider the following three cases: (I) $\alpha + p < \beta + q < k$, (II) $\alpha + p < k \leq \beta + q$ and (III) $k \leq \alpha + p \leq \beta + q$.

Let us prove each case, respectively. Suppose that $AB \neq 0$.

(I) $\alpha + p < \beta + q < k$: Since the monomial $y^\beta z^q$ belongs to $f \circ \phi$ by Theorem 3.6, $y^\beta z^q \in ug$. So $(\beta,q) \geq (\delta,t)$ and then $\delta + t < k$. By Theorem 3.6 again, $(\delta,t) = (\beta,q)$.

(II) $\alpha + p < k \leq \beta + q$: If $\delta + t \geq k$ then by Theorem 3.5 $(\delta,t) = (\beta,q)$. If $\delta + t < k$ then by applying the same method to $u^{-1}f = g \circ \phi^{-1}$, we would have a contradiction by Theorem 3.6.

(III) $k \leq \alpha + p \leq \beta + q$: Since $\delta + t \geq \gamma + s = \alpha + p$, by Theorem 3.4 $(\delta,t) = (\beta,q)$.

Thus we proved that if $f \approx g$ then $(\alpha, p) = (\gamma, s)$ and $(\beta, q) = (\delta, t)$. To prove that $a^n = d^k = a^q d^\beta$ and $Aa^p d^\alpha = a^n B$ for some nonzero numbers $a$ and $d$, it is enough just to compare coefficients of $z^n$, $y^\alpha z^p$, $y^\beta z^q$ and $y^k$ in $f \circ \phi$ and $g$, respectively. Then $a^n = d^k = d^\beta a^q$ and $Ad^\alpha a^p = a^n B$. Also, since $a^{n\beta+kq-nk} = a^{n\beta}a^{k(q-n)} = d^{k\beta}d^{-k\beta} = 1$ and $d^{n\beta+kq-nk} = d^{n(\beta-k)}d^{kq} = a^{-nq}a^{nq} = 1$, $Ad^\alpha a^p = a^n B$ implies that $A^{n\beta+kq-nk} = B^{n\beta+kq-nk}$.

To prove the converse, define a nonsingular mapping $\psi$ by $\psi(z, y) = (az, dy)$ for given numbers $a, d$. $(f \circ \psi)(z, y) = a^n(z^n + d^\alpha a^{p-n} A y^\alpha z^p + d^\beta a^{q-n} y^\beta z^q + y^k) = a^n g(z, y)$. Thus we finished the proof of Theorem 3.7.

Corollary 3.8. *Under the same assumption as in Theorem 3.7, if $A$ and $B$ are complex numbers and $n\beta + kq - nk = 1$ then $f \approx g$ if and only if $(\alpha, p) = (\gamma, s)$, $(\beta, q) = (\delta, t)$ and $A = B$.*

Let $f(z, y) = z^n + A_1 y^{\alpha_1} z^{p_1} + \cdots + A_t y^{\alpha_t} z^{p_t} + y^k$ where $n < k$, $(n, k) = 1$, $n - 2 \geq p_1 > \cdots > p_t \geq 1$, $\frac{\alpha_i}{n-p_i} > \frac{k}{n}$ for $i = 1, \ldots, t$ and each $A_i$ is a unit in ${}_2\mathcal{O}$ for $i = 1, \ldots, t$. Then we may assume without loss of generality that $A_t = 1$ considering $f(\varepsilon^k z, \varepsilon^n y)$ with a suitable unit $\varepsilon$ for the analytic classification.

Theorem 3.9. *Let $f = z^n + A_1 y^{\alpha_1} z^{p_1} + \cdots + A_{t-1} y^{\alpha_{t-1}} z^{p_{t-1}} + y^{\alpha_t} z^{p_t} + y^k$ where $n < k$, $(n, k) = 1$, $n - 2 \geq p_1 > \cdots > p_t \geq 1$, $\alpha_t \leq k - 2$. $\alpha_1 + p_1 < \cdots < \alpha_t + p_t$, $\frac{\alpha_1}{n-p_1} > \cdots > \frac{\alpha_t}{n-p_t} > \frac{k}{n}$ and each $A_i = A_i(z, y)$ is a unit in ${}_2\mathcal{O}$ for $i = 1, \ldots, t-1$. Let $g = z^n + B_1 y^{\beta_1} z^{q_1} + \cdots + B_{s-1} y^{\beta_{s-1}} z^{q_{s-1}} + y^{\beta_s} z^{q_s} + y^k$ where $n - 2 \geq q_1 > \cdots > q_s \geq 1$, $\beta_s \leq k - 2$, $\beta_1 + q_1 < \cdots < \beta_s + q_s$, $\frac{\beta_1}{n-q_1} > \cdots > \frac{\beta_s}{n-q_s} > \frac{k}{n}$ and each $B_j = B_j(z, y)$ is a unit in ${}_2\mathcal{O}$ for $j = 1, \ldots, s - 1$. If $f \approx g$, then $t = s$, $(\alpha_i, p_i) = (\beta_i, q_i)$ for $i = 1, \ldots, t$ and $A_i(0, 0)^{n\alpha_t + kp_t - nk} = B_i(0, 0)^{n\alpha_t + kp_t - nk}$ for $i = 1, \ldots, t - 1$. In particular, if the $A_i$ and the $B_j$ are complex numbers and $n\alpha_t + kp_t - nk = 1$ with the same assumption above, then $f \approx g$ if and only if $(\alpha_i, p_i) = (\beta_i, q_i)$ for $i = 1, \ldots, t = s$ and $A_i = B_i$.*

*Proof.* Use the induction method on $t$ or $s$ and Theorem 3.7.

Remark 3.10. In Theorem 3.7 and Theorem 3.9 we can prove the same result with the following numerical assumption, $k \leq \alpha_1 + p_1 \leq$

$\cdots \leq \alpha_t + p_t$ instead of $\alpha_1 + p_1 < \cdots < \alpha_t + p_t$. If not, the same result may not hold by the example below:

$$z^4 + Ay^6z^2 + y^7z + y^9 \approx z^4 + y^7z + y^9 \quad \text{at the origin}$$

for any number $A$ by Theorem 2.4,(iii).

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