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SOME NECESSARY CONDITIONS FOR ERGODICITY OF NONLINEAR FIRST ORDER AUTOREGRESSIVE MODELS

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1. Introduction

Consider nonlinear autoregressive processes of order 1 defined by the random iteration

$$X_{n+1} = f(X_n) + \epsilon_{n+1} \quad (n \geq 0) \tag{1}$$

where f is real-valued Borel measurable function on R^1, $\{\epsilon_n : n \geq 1\}$ is an i.i.d.sequence whose common distribution F has a non-zero absolutely continuous component with a positive density, $E|\epsilon_n| < \infty$, and the initial X_0 is independent of $\{\epsilon_n : n \geq 1\}$. The process $\{X_n : n \geq 0\}$ is Markovian with (one-step) transition probability

$$p(x, B) := P(f(X) + \epsilon_1 \in B) \quad (x \in R^1,\ B \in \mathcal{B}^1), \tag{2}$$

where P is the probability measure on the underlying probability space (on which X_0, $\{\epsilon_n : n \geq 1\}$ are defined), and $\mathcal{B}^1$ is the Borel σ-field on R^1. It may be noted that all Markov processes on $(R^1, \mathcal{B}^1)$ may be generated by random iterations of the form $X_{n+1} = h(X_n, \epsilon_{n+1})$, where h is a real-valued measurable function on R^2 (See, e.g., Kifer(1986), pp.8, or Bhattacharya and Waymire (1990),pp.228). In our case $h(x, \epsilon) = f(x) + \epsilon$.

A Markov process $\{X_n : n \geq 0\}$, or its transition probability in (2), is said to have an invariant probability π if

$$\int p(x, B)\, \pi(dx) = \pi(B) \quad for\ every\ B \in \mathcal{B}^1. \tag{3}$$

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The process $\{X_n : n \geq 0\}$, or its transition probability $p(x, dy)$, will be said to be irreducible with respect to Lebesgue measure λ on $(R^1, \mathcal{B}^1)$ if $\sum_{n\geq 0} 2^{-n} p^n(x, B) > 0$ for every x and every B with $\lambda(B) > 0$.
Here p^n denotes the n-step transition probability. An irreducible process is recurrant (or, λ -recurrent)if, for every x and every B with $\lambda(B) > 0$,

$$P(X_n \in B \quad \text{for some} \quad n \geq 1 | X_0 = x) = 1.$$

If the latter probability is less than 1 for some B with $\lambda(B) > 0$ and a set A of x such that $\lambda(A) > 0$, then the process is transient. An irreducible process is aperiodic if there do not exist $d > 1$ and disjoint sets $C_1, C_2, \cdots, C_d$ such that $\lambda(C_i) > 0$ and $p(x, C_{i+1}) = 1$ for every $x \in C_i$. (with $C_{d+1} := C_1$), $1 \leq i \leq d$. A λ-recurrent aperiodic process is ergodic,or Harris ergodic,if it has a unique invariant probability π ; in this case

$$(4) \qquad \sup_{B \in \mathcal{B}^1} |p^n(x, B) - \pi(B)| \to 0 \ \ as \ \ n \to \infty, \quad \text{for every} \quad x \in R^1.$$

If the convergence in (4) is exponentially fast then the Harris ergodic process is said to be geometrically (Harris) ergodic.

Lee([3], [4]) provided sets of sufficient conditions for ergodicity and for geometric ergodicity in terms of the quantities

$$(5) \qquad \underline{\alpha} := \underline{\lim}_{x \to -\infty} \frac{f(x)}{x}, \quad \overline{\alpha} := \overline{\lim}_{x \to -\infty} \frac{f(x)}{x},$$

$$(6) \qquad \underline{\beta} := \underline{\lim}_{x \to \infty} \frac{f(x)}{x}, \quad \overline{\beta} := \overline{\lim}_{x \to \infty} \frac{f(x)}{x}.$$

For the special class with

$$(7) \qquad f(x) = \alpha x \mathbf{1}_{\{x<0\}} + \beta x \mathbf{1}_{\{x \geq 0\}}, \ \ E\epsilon_1 = 0,$$

Petrucelli and Woolford[6] proved that '$\alpha < 1, \beta < 1, \alpha\beta < 1$' is necessary as well as sufficient for ergodicity. This is of course not true for the general nonlinear model(1) (for which $\underline{\alpha} = \overline{\alpha}$, $\underline{\beta} = \overline{\beta}$).

In this article, by proving some necessary conditions,we show that sufficient criterion in [4] is nearly necessary.

2. Some Necessary Conditions for Harris ergodicity

Consider the stochastic process $\{X_n : n = 0, 1, 2, 3, \cdots\}$ defined by recursively by

$$X_{n+1} = f(X_n) + \epsilon_{n+1} \quad (n \geq 0) \tag{8}$$

We make the following assumptions: f is real-valued Borel measurable functions on R^1 and continuous, $\{\epsilon_n : n \geq 1\}$ is a sequence of i.i.d.random variables whose common distribution F has a component with an almost everywhere positive absolutely continuous density with respect to Lebesgue measure. Also, $E\epsilon_1 = 0$.
The initial random variable X_0 is independent of $\{\epsilon_n : n \geq 1\}$.

Define

$$\alpha = \lim_{x \to -\infty} \frac{f(x)}{x}, \qquad \beta = \lim_{x \to \infty} \frac{f(x)}{x}. \tag{9}$$

THEOREM 2.1. *Under the conditions on f and $\{\epsilon_n : n \geq 1\}$ specified above and assumption that α and β exist, the Markov process $\{X_n : n \geq 0\}$ is not(Harris) ergodic if one of the following two conditions holds:*

(I) $\alpha > 1$ *or* $\beta > 1$.
(II) $\beta < 0$ *and* $\alpha\beta > 1$.

Proof. First,we prove part I. Without loss of generality,consider the case $\beta > 1 (\beta < \infty)$. Then, for $X_n > 0, n \geq 0, E(X_{n+1}|X_n) = f(X_n)$.

Since $\beta = lim_{x\to\infty} \frac{f(x)}{x}$, there exists a M_η such that $x > M_\eta$ implies

$$x < \eta x < f(x).$$

Thus for any $1 < \eta < \beta$, and $X_n > M_\eta (n \geq 0)$,

$$\begin{aligned} P(X_{n+1} \leq & 2^{-1}(\eta + 1)X_n|X_n) \\ & \leq P(|X_{n+1} - E(X_{n+1}|X_n)| \geq 2^{-1}(\eta - 1)X_n|X_n) \\ & \leq E(|X_{n+1} - E(X_{n+1}|X_n)|X_n) \cdot 2((\eta - 1)X_n)^{-1}. \end{aligned}$$

But

$$X_{n+1} - E(X_{n+1}|X_n) = \epsilon_{n+1}.$$

So we get

$$(10) \qquad P(X_{n+1} \le 2^{-1}(\eta+1)X_n|X_n) \le 2E|\epsilon_1| \cdot [(\eta-1)X_n]^{-1}.$$

Let $c = 2E|\epsilon_1|[(\eta-1)M]^{-1}$. Choose $M > 0$ such that

$$2E|\epsilon_1|[(\eta-1)M]^{-1} < 1.$$

Then, whenever $X_1 > max\{M_\eta, M\}$, (10) implies that

$$P(X_2 \le 2^{-1}(\eta+1)X_1|X_1) \le c,$$

and thus

$$P(X_2 > 2^{-1}(\eta+1)X_1|X_1) \ge (1-c),$$

and

$$\begin{aligned} P(X_3 >& 2^{-1}(\eta+1)X_2, X_2 > 2^{-1}(\eta+1)X_1|X_1) \\ =& E\left[P(X_3 > 2^{-1}(\eta+1)X_2|X_2)\mathbf{1}_{\{X_2>2^{-1}(\eta+1)X_1|X_1\}}\right]. \end{aligned}$$

On the other hand,

$$\begin{aligned} 2E|\epsilon_1|[(\eta-1)X_2]^{-1} \le& 2E|\epsilon_1|[(\eta-1)2^{-1}(\eta+1)X_1]^{-1} \\ =& c \cdot 2(\eta+1)^{-1} \equiv \gamma c. \end{aligned}$$

on the set $\{X_2 > 2^{-1}(\eta+1)X_1, X_1 > M\}$ where $\gamma = 2(\eta+1)^{-1} < 1$. So,

$$P(X_3 > 2^{-1}(\eta+1)X_2, X_2 > 2^{-1}(\eta+1)X_1|X_1) \ge (1-\gamma c)(1-c).$$

Continuing in this manner,whenever $X_1 > max\{M_\eta, M\}$,

$$\begin{aligned} P(X_{\ell+1} > 2^{-1}(\eta+1)X_\ell,\ \ell = 1, 2, \cdots, n|X_1) \ge& \prod_{i=1}^{n}(1 - c\gamma^{i-1}) \\ \ge& (1-c)^{\frac{1}{1-\gamma}} \quad \text{for all} \quad n. \end{aligned}$$

Consequently for any $X_0 \in R^1$,

$$\begin{aligned} P(X_n \to \infty|X_0) \ge& (1-c)^{\frac{1}{1-\gamma}} P(X_1 > \max\{M_\eta, M\}|X_0) \\ >& 0 \end{aligned}$$

Hence, $\{X_n\}$ is not ergodic for $\beta > 1$.

To prove that (II), we prove two lemmas.

LEMMA 2.1. *If* $\beta < -1$, $\alpha\beta > 1$, *then, for* $1 < \eta < \alpha\beta$, *there exists* $M_1 > 0$ *such that* $X_{n-2} > M_1$ *implies*

$$E(X_n|X_{n-2}) \geq \eta X_{n-2}, \quad n \geq 2.$$

Proof.

$$\begin{aligned} E(X_n|X_{n-2}) =& E(f(f(X_{n-2}) + \epsilon_{n-1}) \cdot I_{(f(X_{n-2})+\epsilon_{n-1} \leq 0)}|X_{n-2}) \\ &+ E(f(f(X_{n-2}) + \epsilon_{n-1}) \cdot I_{(f(X_{n-2})+\epsilon_{n-1} > 0)}|X_{n-2}). \end{aligned}$$

For a given $X_{n-2} = x(> 0)$,

$$\begin{aligned} E(X_n|X_{n-2} = x) =& E(f(f(x) + \epsilon_{n-1}) \cdot I_{(f(x)+\epsilon_{n-1} \leq 0)}) \\ &+ E(f(f(x) + \epsilon_{n-1}) \cdot I_{(f(x)+\epsilon_{n-1} > 0)}) \end{aligned}$$

For $1 < \eta < \alpha\beta$, choose $\theta > 0$ such that $\alpha + \theta < 0$, $\beta + \theta < -1$ and $1 < \eta < (\alpha + \theta)(\beta + \theta) < \alpha\beta$.

By our hypotheses, there exist $M_\theta > 0$ and $M_\theta' > 0$ such that

$$0 < (\alpha + \theta)x < f(x) \quad if \ \ x < -M_\theta,$$

and

$$f(x) < (\beta + \theta)x < 0 \quad if \ \ x > M_\theta'.$$

Hence, for $x > M_\theta'$,

$$\begin{aligned} E(f(X_{n-1})|X_{n-2} = x) > & (\alpha + \theta)(\beta + \theta)x \cdot P(f(x) + \epsilon_{n-1} < -M_\theta) \\ &+ E[f(f(x) + \epsilon_{n-1}) \cdot I_{(-M_\theta \leq f(x)+\epsilon_{n-1} \leq 0)}] \\ &+ (\beta - \theta)E|\epsilon_1| \\ &+ E[f(f(x) + \epsilon_{n-1}) \cdot I_{(0 \leq f(x)+\epsilon_{n-1} \leq M_\theta')}] \end{aligned}$$

Let m_1 be the minimum of $f(f(x) + \epsilon_{n-1})$ on S_1 where $S_1 = \{\omega : -M_\theta \leq f(x) + \epsilon_{n-1}(\omega) \leq 0\}$ and m_2 be the minimum of $f(f(x) + \epsilon_{n-1})$ on S_2 where $S_2 = \{\omega : 0 \leq f(x) + \epsilon_{n-1}(\omega) \leq M_\theta'\}$. Then, for $x > M_\theta'$,

$$\begin{aligned} E(f(X_{n-1})|X_{n-2} = x) >& (\alpha + \theta)(\beta + \theta)x \cdot P(f(x) + \epsilon_{n-1} < -M_\theta) \\ &+ m_1 + m_2 + (\beta - \theta)E|\epsilon_1|. \end{aligned}$$

Since $(\alpha + \theta)(\beta + \theta) > \eta$, and $P(f(x) + \epsilon_{n-1} < -M_\theta) \uparrow 1$ as $x \to \infty$, there exists M_1 such that $x > M_1$ implies

$$E(X_n|X_{n-2} = x) = E(f(X_{n-1})|X_{n-2} = x) > \eta x. \quad \square$$

LEMMA 2.2. *Assume that $f'(x)$ exists for sufficiently large values of $|x|$ and is bounded at $\pm\infty$(i.e.,there exists $A > 0$ such that $f'(x)$ exists for $|x| \geq A$ and $\sup_{|x|\geq A}|f'(x)| < \infty$.)*

Then there exists $M_2 > 0$ such that

$$E|f(f(x)+\epsilon_1) - Ef(f(x)+\epsilon_1)| \leq \xi < \infty$$

for some ξ, for all $x > M_2$.

Proof. Choose $\theta > 0$ such that $\beta + \theta < -1$, $\alpha + \theta < 0$. For that θ, there exists $M_\theta > 0$ such that $x > M_\theta$ implies $(\beta - \theta)x < f(x) < (\beta + \theta)x < 0$ and there exists $M_\theta' > 0$ such that $x < -M_\theta$ implies $0 < (\alpha + \theta)x < f(x) < (\alpha - \theta)x$.

Choose M_θ largely enough that $x > M_\theta$ implies

$$f(x) < (\beta + \theta)x < -M_\theta'.$$

For $x > M_\theta$ such that $f(x) + \epsilon_1 < -M_\theta'$,

$$\begin{aligned}|f(f(x)+\epsilon_1) - f(f(x))| \leq & \sup_{x \leq max\{-M_\theta', M_\theta(\beta+\theta)\}} |f'(x)| \cdot |\epsilon_1| \\ \leq & |(\alpha - \theta)||\epsilon_1|,\end{aligned}$$

and thus

$$\begin{aligned}E[\{f(f(x)+\epsilon_1) - &f(f(x))\} \cdot I_{(f(x)+\epsilon_1 < -M_\theta')}] \\ &\leq |(\alpha - \theta)|E|\epsilon_1| \\ &< \infty.\end{aligned}$$

For $x > M_\theta$ such that $f(x) + \epsilon_1 \geq -M_\theta'$,

$$\begin{aligned}&E[\{f(f(x)+\epsilon_1) - f(f(x))\} \cdot I_{(f(x)+\epsilon_1 \geq -M_\theta')}] \\ \leq &C_1 + (\beta - \theta)^2 x \cdot P(f(x)+\epsilon_1 > M_\theta) + |(\beta - \theta)|E|\epsilon_1| \\ &+ |(\alpha - \theta)||(\beta - \theta)|x \cdot P(f(x)+\epsilon_1 \geq -M_\theta') \quad \text{for some} \quad C_1 \\ \leq &C_1 + (\beta - \theta)^2 x \cdot \frac{E|\epsilon_1|}{|-f(x)+M_\theta|} + |(\beta - \theta)|E|\epsilon_1| \\ &+ |(\alpha - \theta)||(\beta - \theta)| \cdot \frac{E|\epsilon_1|}{|-f(x)-M_\theta'|}\end{aligned}$$

$$\leq C_1 + (\beta-\theta)^2 x \cdot \frac{E|\epsilon_1|}{|\beta+\theta|x}$$
$$+ |(\beta-\theta)|E|\epsilon_1| + |(\alpha-\theta)||(\beta-\theta)|x \cdot \frac{E|\epsilon_1|}{|-(\beta+\theta)x - M'_\theta|}.$$

Since $\lim_{x\to\infty} \frac{x}{-(\beta+\theta)x-M'_\theta} = -\frac{1}{\beta+\theta} < \theta_0 < 1$ for some θ_0, there exists M^* such that

$$x > M^* \quad \text{implies} \quad \frac{x}{-(\beta+\theta)x - M'_\theta} < \theta_0 < 1.$$

Let $M_2 = \max\{M_\theta, M^*\}$, then $x > M_2$ implies, for some ξ,

$$2\cdot E|f(f(x)+\epsilon_1) - f(f(x))| \leq \xi < \infty$$

Proof of Theorem 2.1(II). Let M be a number such that

$$M > \max\{M_1, M_2\} \ \textit{and} \ c = 2\xi[(\eta-1)M]^{-1} < 1.$$

For any $1 < \eta < \alpha\beta$ and $X_{2(n-1)} > M, \quad n \geq 1,$

$$\begin{aligned}
&P(X_{2n} \leq 2^{-1}(\eta+1)X_{2(n-1)}|X_{2(n-1)})\\
\leq\ & \text{(by lemma 2.1)}\\
\leq\ & P(X_{2n} - E(X_{2n}|X_{2(n-1)})\\
\leq\ & 2^{-1}(\eta+1)X_{2(n-1)} - \eta X_{2(n-1)}|X_{2(n-1)})\\
=\ & P(X_{2n} - E(X_{2n}|X_{2(n-1)}) \leq -2^{-1}(\eta-1)X_{2(n-1)}|X_{2(n-1)})\\
\leq\ & P(|X_{2n} - E(X_{2n}|X_{2(n-1)})| \leq 2^{-1}(\eta-1)X_{2(n-1)}|X_{2(n-1)})\\
\leq\ & E(|X_{2n} - E(X_{2n}|X_{2(n-1)})||X_{2(n-1)})\cdot 2[(\eta-1)M]^{-1}\\
\leq\ & 2\xi\cdot[(\eta-1)M]^{-1} < 1 \qquad \text{(by lemma 2.2)}
\end{aligned}$$

Then, whenever $X_2 > M$,

$$P(X_4 > 2^{-1}(\eta+1)X_2|X_2) \geq 1 - c.$$

$$\begin{aligned}
&P(X_6 > 2^{-1}(\eta+1)X_4, X_4 > 2^{-1}(\eta+1)X_2|X_2)\\
&\quad = P(X_6 > 2^{-1}(\eta+1)X_4|X_4 > 2^{-1}(\eta+1)X_2, X_2)\cdot\\
&\qquad P(X_4 > 2^{-1}(\eta+1)X_2|X_2)\\
&\quad = P(X_6 > 2^{-1}(\eta+1)X_4|X_4 > 2^{-1}(\eta+1)X_2)\cdot\\
&\qquad P(X_4 > 2^{-1}(\eta+1)X_2|X_2)\\
&\qquad\qquad (\text{because } \{X_{2n}, n \geq 0\} \text{ is a Markov process.})\\
&\quad \geq (1 - c[2^{-1}(\eta+1)]^{-1})\cdot(1-c) = (1-\gamma c)(1-c)\\
&\qquad \text{where } \gamma = 2(\eta+1)^{-1} < 1
\end{aligned}$$

Continuing in this manner, whenever $X_2 > M$,

$$\begin{aligned}
P(X_{2(\ell+1)} &> 2^{-1}(\eta+1)X_{2\ell},\ \ell = 1, 2, \cdots, n|X_2)\\
&\geq \prod_{i=1}^{n}(1 - c\gamma^{i-1}) \geq (1-c)^{\frac{1}{1-\gamma}}
\end{aligned}$$

For any $x_0 \in R^1$, $P(X_2 > M|X_0) > 0$, and thus

$$P(X_{2n} \to \infty|X_0) \geq (1-c)^{\frac{1}{1-\gamma}} \cdot P(X_2 > M|X_0) > 0.$$

Hence, $\{X_{2n}\}$ is not ergodic, neither is $\{X_n\}$. □

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