J. Korean Math. Soc. **33** (1996), No. 1, pp. 169–181

SOME LINEARLY INDEPENDENT IMMERSIONS INTO THEIR ADJOINT HYPERQUADRICS

CHANGRIM JANG

1. Introduction

Let $x : M^n \longrightarrow E^m$ be an isometric immersion of an n-dimensional connected Riemannian manifold into the m-dimensional Euclidean space. Then the metric tensor on M^n is naturally induced from that of E^m. We use the same notation $<,>$ for the metrics and identify M^n with $x(M^n)$ unless stated ortherwise. Let ∇ and $\overline{\nabla}$ be the Levi-Civita connections on M^n and E^m respectively. Then, we have so-called Gauss equation $\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$, where X and Y denote vector fields on M^n and h is the second fundamental form. The equation of Weingarten is given by $\overline{\nabla}_X \xi = -S_\xi X + D_X \xi$, where S_ξ is the Weingarten map associated with a normal vector field ξ to M^n and D the normal connection in the normal bundle NM. Denote by Δ the Laplacian of M^n. Then the following two equations are well known:

$$\Delta x = tr\, h = \sum_{i=1}^{n} h(e_i, e_i) \tag{1.1}$$

for a local orthonormal frame $e_1, e_2, \dots, e_n$ of M^n;

$$\Delta\langle x, x\rangle = 2\langle \Delta x, x\rangle + 2n. \tag{1.2}$$

The immersion x is of finite type k[1] if the position vector field x admits the following decomposition

$$x = x_0 + x_1 + \cdots + x_k, \tag{1.3}$$

Received December 26, 1994.

1991 AMS Subject Classification: 53C40, 53A05.

Key words and phrases: Finite type immersions, linearly independent immersions, orthogonal immersions, adjoint hyperquadrics.

where x_0 is a constant vector in E^m and $\Delta x_i = \lambda_i x_i$; $\lambda_i \in R$ and $\lambda_i \neq \lambda_j$ if $i \neq j$. For a k- type immersion x, let $E_i, i = \in \{1, 2, \ldots, k\}$, denote the subspace of E^m spanned by $x_i(p); p \in M^n$. The immersion x is said to be linearly independent if the linear subspace $E_1, E_2, \ldots, E_n$ are linearly independent. It is said to be an orthogonal immersion if $E_1, E_2, \ldots, E_k$ are mutually orthogonal.[2,3] Linearly independent immersions and orthogonal immersions are strongly related with the immersions satisfying the equation

$$\Delta x = Ax + b \tag{1.4}$$

for some constant $m \times m$ matrix A and some constant vector b in E^m. In [5] O.J.Garay initiated to study hypersurfaces satisfying (1.4) for some diagonal matrix A and $b = 0$. And several authors studied linearly independent immersions, orthogonal immersions and immersions satisfying kinds of the equation (1.4). The followings are some important results concerned with these objects:

THEOREM A. [2] *Let $x : M^n \longrightarrow E^m$ be an immersion of finte type. Then the immersion x is linearly independent if and only if x satisfies (1.4) for some constant matrix $A \in E^{m\times m}$ and $b \in E^m$.*

THEOREM B. [2] *Let $x : M^n \longrightarrow E^m$ be an immersion of finte type. Then the immersion x is orthogonal if and only if x satisfies (1.4) for some symmetric matrix $A \in E^{m\times m}$ and $b \in E^m$.*

THEOREM C. [2,4,6] *Let $x : M^n \longrightarrow E^{n+1}$ be an isometric immersion satisfying (1.4) for some A and b. Then M^n is minimal in E^{n+1} or an open part of an hypersphere or an open part of a spherical cylinder.*

THEOREM D. [4] *Let $x : M^n \longrightarrow S_0^{n+1}(r) \subset E^{n+2}$ be an isometric immersion into the $(n+1)$-dimensional sphere with radius r centered at origin, satisfying (1.4) for some A and $b = 0$. Then M^n is minimal in $S_0^{n+1}(r)$ or an open part of an n-dimensional sphere or an open part of a product of two spheres.*

Let $x : M^n \longrightarrow E^m$ be a linearly independent immersion whose spectral decomposition is given by (1.3). Let $u = (u_1, u_2, \ldots, u_m)$ be a Euclidean coordinate system on E^m with x_0 as its origin. Then one can show that there exists an $m \times m$ matrix A satisfying $\Delta x = Ax$, following the process in the proof of theorem A. And if x is nonminimal and is fully contained in E^m(i.e., M^n is not contained in a hyperplane of E^m.), the non-zero matrix A is uniquely defined. Now we can introduce the notion of adjoint hyperquadrics.

DEFINITION. [3] Let $x : M^n \longrightarrow E^m$ be a non-minimal, linearly independent and fully contained immersion. Let A be the $m \times m$ matrix associated the immersion x defined above. Then for any point p in M^n, the equation

$$\langle Au, u\rangle = \sum_{i,j}^{m} a_{ij}u_iu_j = c_p(c_p = \langle Ax, x\rangle(p))$$

defines a hyperquadric Q_p in E^m. We call the hyperquadric Q_p the adjont hyperquadric of the immersion x at p. In paticular, if M^n is contained in an adjoint hyperquadric Q_p of x for some point $p \in M^n$, then all of the adjoint hyperquadrics $\{Q_p | p \in M^n\}$ give a common adjoint hyperquadric, denoted by Q. We call the hyperquadric Q the adjoint hyperquadric of the linearly independent immersion x.

The following theorem provides a necessary and sufficient condition for a linearly immersion to be an orthogonal immersion in terms of the adjoint hyperquadrics.

THEOREM E. [3] *Let $x : M \longrightarrow E^m$ be a non-minimal linearly independent immersion. Then M is immersed by x as a minimal submanifold of its adjont hyperquadric if and only if the immersion x is an orthogonal immersion*

In this paper, we investigate some linearly independent immersions with lower codimensions and observe that the necessary condition for TheoremE can be weakened in these cases. Our results are as follows:

THEOREM 1. *Let $x : M^n \longrightarrow E^{n+2}$ a non-minimal fully contained linearly independent immersion. Then M^n is immersed into its adjont hyperquadric if and only if the immersion x is an orthogonal immersion.*

THEOREM 2. *Let $x : M^n \longrightarrow E^{n+3}$ a fully contained linearly independent immersion and let M^n be a compact manifold. Then M^n is* *immersed into its adjont hyperquadric if and only if the immersion x is an orthogonal immersion.*

Continuously, we investigate some spherical immersions satisfying kinds of the equation (1.4) and obtain the followings which are generalizations of Theorem D.

THEOREM 3. *Let $x : M^n \longrightarrow S_a^{n+1}(r) \subseteq E^{n+2}$ be an isometric immersion into an $(n+1)$-dimensional sphere centered at a point a in E^{n+3} and let x satisfy (1.4) for some constant matrix $A \in E^{(n+2)\times(n+2)}$ and $b \in E^{n+2}$. Then M^n is one of the followings:*

(1) *a minimal hypersurface of $S_a^{n+1}(r)$;*
(2) *an open part of an n-dimensional sphere;*
(3) *an open part of a product of two spheres $S^p(r_1) \times S^{n-p}(r_2)$,where $p = 1, 2, \ldots, n-1$. $r_1^2 + r_2^2 = r^2$ and $\frac{p}{r_1^2} \neq \frac{n-p}{r_2^2}$.*

THEOREM 4. *Let $x : M^n \longrightarrow S_0^{n+2}(r) \subseteq E^{n+3}$ be an isometric immersion and x satisfy (1.4) for some constant matrix $A \in E^{(n+3)\times(n+3)}$ and $b = 0$. Then x is an orthogonal immersion.*

2. Proofs of Theorem1 and Theorem2

Let $x : M^n \longrightarrow E^m$ be an isometric immersion satisfying the equation (1.4) for some A and b. Let C denote the skew symmetric matrix $\frac{1}{2}(A - {}^tA)$. Then we have the following lemmas.

LEMMA 2.1. *For every point p of M^n and orthonormal tangent vectors $e_1(p), e_2(p), \ldots, e_n(p)$ at p, we have*

$$\sum_{i=1}^{n} \langle Ae_i(p), e_i(p) \rangle = -\langle Ax(p) + b, Ax(p) + b \rangle.$$

Proof. Let $e_1, e_2, \ldots, e_n$ be a local orthonormal frame of M^n defined on U. Then we have

$$\langle Ax + b, e_i \rangle = 0$$

for $i = 1, 2, \ldots, n$. Differentiating the above equations in the direction of e_i, we get

$$\langle Ae_i, e_i \rangle + \langle Ax + b, h(e_i, e_i) \rangle = 0.$$

By summation the following holds

$$\sum_{i=1}^{n}\langle Ae_i, e_i\rangle + \langle Ax + b, \sum_{i=1}^{n} h(e_i, e_i)\rangle = 0.$$

Combining this with (1.1), we get the conclusion.

LEMMA 2.2. *For every tangent vector field X of M^n and the skew symmetric matrix C, the vector field CX is normal to M^n.*

Proof. Let X, Y be local tangent vector fields on M^n. Then we have

$$\langle Ax + b, Y\rangle = 0.$$

Differentiaing this equation in the direction X, we get

$$\langle AX, Y\rangle + \langle Ax + b, h(X, Y)\rangle = 0.$$

Since h is symmetric, we get $\langle AX, Y\rangle = \langle AY, X\rangle$. Thus $\langle CX, Y\rangle = 0$.

LEMMA 2.3. *If x satisfies the eqation $\langle Ax + b, x\rangle = c$ for some constant c, then $2Cx + b$ is normal to M^n and $\langle Ax + b, 2Cx + b\rangle = 0$.*

Proof. Let X be a local tangent vector field of M^n. Differentiating the eqation $\langle Ax + b, x\rangle = c$ in the direction of X, we get

$$\langle AX, x\rangle + \langle Ax + b, X\rangle = 0$$

This implies $\langle {}^tAx, X\rangle = 0$ and $\langle 2Cx + b, X\rangle = 0$. Let $e_1, e_2, \ldots, e_n$ be a local orthonormal frame of M^n. Then we have $\langle {}^tAx, e_i\rangle = 0$, $i = 1, 2, \ldots, n$. By differentiating this formula in the direction e_i, we get $\langle {}^tAe_i, e_i\rangle + \langle {}^tAx, \overline{\nabla}_{e_i} e_i\rangle = 0$. Hence the following holds

$$\sum_{i=1}^{n}\langle {}^tAe_i, e_i\rangle + \langle {}^tAx, Ax + b\rangle = 0.$$

Since $\langle {}^tAe_i, e_i\rangle = \langle Ae_i, e_i\rangle$, From the above eqation and Lemma2.1 we have that $\langle {}^tAx, Ax + b\rangle = \langle Ax + b, Ax + b\rangle$. This implies $\langle 2Cx + b, Ax + b\rangle = 0$.

Proof of Theorem 1. Let $x : M^n \longrightarrow E^{n+2}$ be a linearly independent immersion into its adjoint hyperquadric. Then there exists a matrix A and a constant c such that $\Delta x = Ax$ and $\langle Ax, x\rangle = c$. Assume the immersion x is not orthogonal. This assumption implies that the skew symmetric matrix $C = \frac{1}{2}(A -{}^t A)$ is not a zero matrix. We proceed with two cases seperately.

Case 1. $\langle Ax, x\rangle = c \neq 0$

In this case we know that Ax is never zero and the set $U = \{p \in M^n | Cx(p) \neq 0\}$ is an open dense subset of M^n. (If Cx is identically zero on an open subset V of M^n, then V is contained in $kerC$ which is a linear subspace of E^{n+2}. Since $rankC$ is at least 2, the dimension of $kerC$ is at most n. This implies V is contained an n dimensional linear subspace of E^{n+2}. Then V is minimal in E^{n+2} and hence $Ax = 0$ on V.) Let U_1 be a connected component of U. Then from Lemma2.3 we know that the vector fields Ax, Cx span the normal space of U_1 and they are mutually orthogonal. Let x^N be the orthogonal projection of x to the normal space. Then we have

$$x^N = \alpha Ax + \beta Cx$$

for some differentiable funtions α and β. But $\langle Cx\ x\rangle = 0$ implies that $\beta = 0$. Hence we know that

$$x = \alpha Ax + x^T, \tag{2.1}$$

where x^T is the tangential part of x. For a local tangent vector field X, we get $\langle Cx, X\rangle = 0$ and hence $\langle CX, x\rangle = 0$. Lemma2.2, (2.1) and the equation $\langle CX, x\rangle = 0$ implies that $CX = \gamma Cx$ for some differntiable funtion γ. Consider the unit normal vector field $\frac{Cx}{|Cx|}$ and take its covariant derivative in the direction of X, we get

$$X(\frac{1}{|Cx|})Cx + \frac{1}{|Cx|}CX = \{X(\frac{1}{|Cx|}) + \frac{\gamma}{|Cx|}\}Cx.$$

This vector field must be orthogonal to Cx. Hence we get $\overline{\nabla}_X \frac{Cx}{|Cx|} = 0$. This means $\frac{Cx}{|Cx|}$ is a constant vector field . Therefore $Cx = |Cx|E$ for some constant vetor E. Let N be a local unit normal vector of U

which is normal to Ax. Then above argument implies every covariant derivative of N vanishes. Let N' be a local unit normal vector field orthogonal to Ax in a neighborhood of $p \in M^n - U$. Then by continuity every covariant derivative of N' must vanishes at p. So we have a global constant normal vector field of M^n. This means M^n is contained a hyperplane of E^{n+2}, which leads a contradiction to the assumption that M^n is fully contained. Thus we must have $C = 0$. And we can conclude that x is an orthogonal immersion.

Case 2. $\langle Ax, x \rangle = 0$

Since x is non-minimal, we may assume $Ax \neq 0$ locally. In this case we can see that $Cx \neq 0$ and Ax, Cx span the normal space of M^n. Since $\langle Ax, x \rangle = 0$ and $\langle Cx, x \rangle = 0$,$x$ is tangential. The integral curve of x is can be expressed as $x(s) = e^s a$ for a constant vector a which is a part of a ray from origin. On this integral curve, Aa and Ca span the normal space of M^n. Hence we know that every point of this integral curve has parallel tangent space. Appealing to lemma 2.1 we get

$$\langle Ax(s), Ax(s) \rangle = constant.$$

This is a contradiction. So we must have $C = 0$. This implies x is an orthogonal immersion. The converse follows from theoremE.

Proof of Theorem 2. Let $x : M^n \longrightarrow E^{n+3}$ be a linearly independent immersion of an n-dimensional compact manifold into its adjoint hyperquadric. Then there exist a matrix A and a constant c such that $\Delta x = Ax$ and $\langle Ax, x \rangle = c$. From (1.2) and $\langle x, Ax \rangle = c$ we can see that $\Delta \langle x, x \rangle = 2c + 2n$. By Hopf's lemma we get $\langle x, x \rangle = r^2$ for some positive constant r and $\langle Ax, x \rangle = -n$. Hence M^n is immersed into $S_0^{n+2}(r)$ by x. By theorem4 which will be proved in the next section we know that A is symmetric and hence x is an orthogonal immersion. The converse follows from theoremE.

3. Proofs of Theorem 3 and Theorem 4

Proof of Theorem 3. Since M^n is contained in $S_a^{n+1}(r)$, we know that $\langle x - a, x - a \rangle = r^2$. Instead of the immersion x, Consider the immersion $y : M^n \longrightarrow E^{n+2}$ given by $y = x - a$. Then M^n is immersed into $S_0^{n+1}(r)$ by y and the following holds

$$\Delta y = \Delta x = Ay + b'.(b' = Aa + b)$$

So without loss of generality we may assume the manifold M^n is immersed into $S_0^{n+1}(r)$. In this case it follows from the eqation (1.2) and $\Delta\langle x, x\rangle = 0$ that $\langle Ax + b, x\rangle = -n$. And hence we get $2Cx + b$ is a normal vector field of M^n and $\langle 2Cx + b, Ax + b\rangle = 0$(by Lemma2.3). Suppose $2Cx + b = 0$ in an open subset V of M^n. Then we have $\langle 2Cx + b, x\rangle = 0$. And hence

(3.1) $$\langle x, b\rangle = 0 \text{ in } V.$$

Then $\Delta\langle x, b\rangle = \langle \Delta x, b\rangle = \langle Ax + b, x\rangle = 0$. This implies

(3.2) $$\langle x, {}^t Ab\rangle = -\langle b, b\rangle.$$

From (3.1) and (3.2) we can conclude $C = 0$ and $b = 0$(Otherwise V is a minimal submanifold of E^{n+2}). In this case referring to theoremD, we get the desired classification. If b or C is nonzero , then the above argument implies the set $\{p \in M^n | 2Cx(p) + b \neq 0\}$ is a dense open subset of M^n. Hence we may assume that $Ax + b$ and $2Cx + b$ are local normal vectors of M^n which are mutually orthogonal. Since x is also a normal vector field of M^n, there exist differentiable funtions α, β such that

$$x = \alpha(Ax + b) + \beta(2Cx + b).$$

Let $e_1, e_2, \ldots, e_n$ be a local orthonormal frame of M^n. Differentiating above equations in the direction e_i, we get

$$e_i = (e_i\alpha)(Ax + b) + \alpha Ae_i + (e_i\beta)(2Cx + b) + \beta 2Ce_i.$$

Hence we get

$$\delta_{ij} = \langle e_i, e_j\rangle = \alpha\langle Ae_i, e_j\rangle$$

This means that the shape operator associated with $Ax + b$ is $-\frac{1}{\alpha}I$, where I is the identity transformation. From Lemma2.2 , we know that the shape operator associated with $2Cx + b$ is the zero map. This argument implies that the set $\{p \in M^n | 2Cx + b \neq 0\}$ is a totally umbilical submanifold. By continuity we can conclude M^n is an open part of n-dimensional sphere.

For the proof of theorem4 we need following two lemmas. We assume the radius r of $S_0^{n+2}(r)$ is 1 for simplicity.

LEMMA 3.1. *If the skew symmetric matrix* $C = \frac{1}{2}(A - {}^t A)$ *is a nonzero matrix and* $Ax \neq -nx$ *on* M^n *, then* M^n *is contained in a hyperplane of* E^{n+3}.

Proof. If $Cx = 0$ on the open subset V of M^n, then V is contained in a $(n+1)$ dimensional linear subspace of E^{n+3}. Hence V is an open part of an n dimensional sphere centered at origin. So $Ax = -nx$ holds on V. This is a contradiction. So we know that the set $U = \{p \in M^n | Cx \neq 0\}$ is an open dense subset of M^n and, $x, Ax + nx$ and Cx are normal vectors in U which are mutually orthogonal from (1.2) and Lemma 2.3. We will only show that in every component of U, the equation holds $Cx = \mu E$ for some constant vector $E \in E^{n+3}$ and a funtion μ. Then by similar argument to that of proof of theorem1 we can see that M^n is contained in a hyperplane of E^{n+3}. Let $e_1, e_2, \dots, e_n$ be a local orthonormal frame of U. Then we have from Lemma2.2 and $\langle Ce_i, x\rangle = 0$

$$Ce_i = \alpha_i(Ax + nx) + \beta_i Cx, i \in \{1, 2, \dots, n\} \tag{3.3}$$

for some differentiable α_i and β_i. If one of α_i is nonzero, without loss of generality we may assume $\alpha_1 \neq 0$. Then we have from (3.3)

$$C(e_2 - \frac{\alpha_2}{\alpha_1}e_1) = (\beta_2 - \frac{\alpha_2}{\alpha_1}\beta_1)Cx$$

$$\vdots$$

$$C(e_n - \frac{\alpha_n}{\alpha_1}e_1) = (\beta_n - \frac{\alpha_n}{\alpha_1}\beta_1)Cx.$$

Let $e'_2, \dots, e'_n$ be local orthonormal vectors spanned by $e_2 - \frac{\alpha_2}{\alpha_1}e_1, \dots, e_n - \frac{\alpha_n}{\alpha_1}e_1$. Then we have a new orthonormal frame $e'_1, e'_2, \dots, e'_n$ such that

$$Ce'_1 = \alpha'_1(Ax + nx) + \beta'_1 Cx$$
$$Ce'_i = \beta'_i Cx, \quad i = 2, 3, \dots, n$$

for some differentiable functions α'_1, β'_i. We will use the notations e_i, α_1, β_i instead of $e'_i, \alpha'_1, \beta'_i$. So the following holds for e_i, α_1, β_i,

$$Ce_1 = \alpha_1(Ax + nx) + \beta_1 Cx, \quad \alpha_1 \neq 0 \tag{3.4}$$

$$Ce_i = \beta_i Cx, \quad i = 2, 3, \dots, n. \tag{3.5}$$

From (3.5) we get

$$C\overline{\nabla}_{e_1}e_2 = (e_1\beta_2)Cx + \beta_2 Ce_1.$$

The right hand side of this is normal, hence

$$C\overline{\nabla}_{e_1}e_2 = Ch(e_1, e_2) + C\nabla_{e_1}e_2$$

is normal. So we have $Ch(e_1, e_2)$ is normal. Let $h(e_1, e_2) = k_1 x + k_2(Ax + nx) + k_3 Cx$ for some funtions k_1, k_2 and k_3. But we see that $k_1 = k_3 = 0$ from simple calculations. So we have $Ch(e_1, e_2) = k_2 C(Ax + nx)$. From (3.4) we get $\langle C(Ax + nx), e_1\rangle \neq 0$. So we must have $k_2 = 0$. This implies $\overline{\nabla}_{e_1}e_2$ is tangential. Similar computations imply the following general facts:

(3.6) $$h(e_i, e_j) = 0 \quad \text{for} \quad i, j = 1, \dots, n \quad \text{and} \quad i \neq j;$$

(3.7) $$h(e_i, e_i) = -x \quad \text{for} \quad i = 2, \dots, n;$$

(3.8) $$\langle \overline{\nabla}_{e_i}e_j, e_1\rangle = 0, \quad \text{thus} \quad \overline{\nabla}_{e_i}e_1 = 0, i, j = 2, \dots, n.$$

From (3.7) and (1.1) we have

(3.9) $$h(e_1, e_1) = Ax + (n-1)x.$$

This implies $\langle Ax, h(e_1, e_1)\rangle = \langle Ax, Ax\rangle - n(n-1)$. Thus

$$\langle Ax, Ax\rangle = -\langle Ae_1, e_1\rangle + n(n-1).$$

Differentiating this equation in the direction e_i for $i = 2, \dots, n$, we have

$$2\langle Ax, Ae_i\rangle = -\langle A\overline{\nabla}_{e_i}e_1, e_1\rangle - \langle \overline{\nabla}_{e_i}e_1, Ae_1\rangle.$$

By (3.8), the right hand side of this equation is zero. So we have

(3.10) $$\langle Ax, Ae_i\rangle = 0 \quad \text{for} \quad i = 2, \dots, n.$$

Since $\langle Ae_i, x\rangle = 0$ and $\langle Ae_i, Cx\rangle = -\langle Ax, Ce_i\rangle = 0$ for $i = 2, \dots, n$, (3.10) implies

(3.11) $$D_{e_i}Ax = 0 \quad \text{for} \quad i = 2, \dots, n.$$

Now we will prove that $\nabla_{e_1} e_1 = 0$. By coddazi equation, we have

$$(\nabla_{e_i} h)(e_1, e_1) = (\nabla_{e_1} h)(e_i, e_1) \quad \text{for} \quad i = 2, \dots, n$$

By (3.6),(3.8),(3.9) and this equation, we obtain

$$D_{e_i} Ax = -h(\nabla_{e_1} e_i, e_1) - h(e_i, \nabla_{e_1} e_1), \quad i = 2, \dots, n.$$

From which and (3.11) we get

$$-h(\nabla_{e_1} e_i, e_1) - h(e_i, \nabla_{e_1} e_1) = 0, \quad i = 2, \dots, n.$$

From this and (3.6), we find

$$-\langle \nabla_{e_1} e_i, e_1 \rangle h(e_1, e_1) - \langle \nabla_{e_1} e_1, e_i \rangle h(e_i, e_i) = 0, \quad i = 2, \dots, n.$$

This implies

$$\nabla_{e_1} e_1 = 0. \tag{3.12}$$

(3.8) and (3.12) implies $D_1 = span\{e_1\}$ and $D_2 = span\{e_2, \dots, e_n\}$ are two complementary integrable distributions. Hence there exists a local coordinate $(s, x_2, \dots, x_n)$ of U such that $e_1 = \frac{\partial}{\partial S}$ and $span\{e_2, \dots, e_n\}$ $= span\{\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\}$. Hence from (3.7),(3.8),(3.9),(3.11) and (3.12) we have

$$\overline{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} = Ax + (n-1)x, \overline{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial s} = 0, \tag{3.13}$$

$$A \frac{\partial}{\partial x_i} = -n \frac{\partial}{\partial x_i}. \tag{3.14}$$

Since the curvature tensor R of E^{n+3} is identically zero, we get from (3.13)

$$\begin{aligned} 0 = R(\frac{\partial}{\partial s}, \frac{\partial}{\partial x_i}) \frac{\partial}{\partial s} &= \overline{\nabla}_{\frac{\partial}{\partial s}} \overline{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial s} - \overline{\nabla}_{\frac{\partial}{\partial x_i}} \overline{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} \\ &= -\overline{\nabla}_{\frac{\partial}{\partial x_i}} \{Ax + (n-1)x\} \\ &= -A \frac{\partial}{\partial x_i} - (n-1) \frac{\partial}{\partial x_i}. \end{aligned}$$

So we get

$$A \frac{\partial}{\partial x_i} = -(n-1) \frac{\partial}{\partial x_i}.$$

This is a contradiction to (3.14). Therefore we must have $\alpha_1 = 0$ in (3.4). Consequently there exist a constant vector E and a funtion μ such that $Cx = \mu E$.

LEMMA 3.2. *Let U_1 be the set $\{p \in |Ax(p) = -nx(p)\}$. If U_1 has nonempty interior and C is an nonzero matrix, then M^n satisfies $Ax = -nx$ globally. Hence M^n is a minimal submanifold of $S_0^{n+2}(1)$.*

Proof. Suppose $U_2 = \{p \in M^n | Ax(p) \neq -nx(p)\}$ is nonempty. If $Cx = 0$ on an open subset V of U_2, then V is contained in $kerC$. Since the dimension of $kerC$ is at most $n+1$, V is contained in an $(n+1)$-dimensional linear subspace of E^{n+3}. Subsequently V is contained in an n-dimensional sphere centered at origin. This implies $Ax = -nx$ on V. Hence the set $W = \{p \in U_2 | Cx(p) \neq 0\}$ is an open dense subset of U_2. By theorem3 and lemma3.1 we know that W is locally an n-dimensional sphere of which center is not origin or a product of two spheres. In every case the vector field Ax is parallel in W and by some choices of a local orthonormal frame $e_1, e_2, \dots, e_n$, we know that

$$Ae_i = k_i e_i \tag{3.15}$$

for some constants $k_i \neq -n$, $i = 1, 2, \dots, n$. The assumption U_1 has nonempty interior implies A has at least an $(n+1)$ dimensional eigen space with $-n$ as its eigen value. And (3.15) implies that A has at least $n+1$ eigen values (counting with multiplicities) different from $-n$. This is a contradiction. So we can conclude that U_2 is empty.

Proof of Theorem 4. If $C = 0$, then the conclusion holds directly. Suppose $C \neq 0$ and $Ax \neq -nx$ at some point $p \in M^n$. Then the set $U = \{p \in M^n | Ax(p) \neq -nx(p)\}$ is an open dense subset of M^n by lemma3.1. In this case, lemma3.1 implies that every component of U is contained in a hyperplane. By theorem3 we can see that the vector field Ax is parallel on U. And thus $\langle Ax, Ax \rangle$ is constant on M^n by continuity. Since this value is different from n^2, we see that $U = M^n$. Appealing to lemma3.1 and theorem3 again we can get the conclusion.

References

1. B. Y. Chen, *Total mean curvature and submanifold of Finite Type*, World Scientific Publisher, 1984.
2. B. Y. Chen and M.Petrovic, *On spectral decomposition of Immersions of Finite Type*, Bull. Austral. Math. Soc. **44** (1991), 117–129.
3. B. Y. Chen, *Linearly Independent, Orthogonal and equivariant immersions*, Kodai J. Math. **14** (1991), 341–349.

4. J. Park, *Geometric and Analytic Characterizations of Isoparpmetic submanifolds*, thesis, Brandeis University.
5. O. J. Garay, *An extension of Takahashi*, Geomatriae Dedicata **34** (1990), 105–112.
6. T. Hasanis and T. Vlachos, *Hypersurfaces of E^{n+1} satisfying $\Delta x = Ax + b$*, J. Austral. Math. Soc. **53** (1992), 377–384.

Department of Mathematics
University of Ulsan
Ulsan 680-749, Korea