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ON THE VOLUME FOR TOTALLY REAL SUBMANIFOLDS OF A KAEHLER MANIFOLD

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1. Introduction

One of the fundamental problems in Riemannian geometry is "How the geometric invariants of Riemannian manifolds are influenced by the curvature restriction ?".

In this note we shall study the volume of a totally real submanifold P of a Kaehler manifold M and obtain an inequality for the relative volume Vol(M)/Vol(P).

Now we recall the definition of n-mean curvature. Let $(M, < , >)$ be a Riemannian manifold of dimension m with metric $< , >$. Let $X_1, \cdots, X_n, Y$ be orthonormal vectors on M and Π the n-plane spanned by $X_1, \cdots, X_n$. The n-mean curvature of Y and Π is defined by

$$K(Y, \Pi) := \sum_{i=1}^{n} < R(Y, X_i)X_i, Y >,$$

where we adopt the following definition for the curvature and the Riemannian Christoffel tensor on M;

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$
$$R(X,Y,Z,W) =< R(X,Y)Z, W > .$$

A submanifold P of an almost complex manifold (M, J) is called totally real provided that the almost complex structure J of M maps

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tangent vectors to P into normal vectors. Let P be an n-dimensional totally real submanifold of a Kaehler manifold (M, J) of real dimension $2n$. Let N be a unit vector field normal to P defined on an open U of P. Let L be the Weingarten map of P associated to N. We define the JN-normal curvature of P at $p \in U$, k_{JN}, as the normal curvature of P at p in the direction JN i.e., $k_{JN}(p) = < LJN, JN > (p)$. We define the JN-mean curvature of P at p by

$$H_{JN}(p) = \frac{1}{n-1}(trL - k_{JN})(p) = \frac{nH - k_{JN}}{n-1}(p),$$

where $H(p)$ is the mean curvature of P at p.

We denote by $CP^n(\lambda)$ the complex projective space of real dimension $2n$ and holomorphic sectional curvature 4λ, and $RP^n(\lambda)$ the real projective space with constant sectional curvature λ. It is well known that there is a natural embedding of real projective space $RP^n(\lambda)$ as totally real, totally geodesic submanifold of $CP^n(\lambda)$.

The main result we shall prove is

THEOREM. *Let M be a connected compact Kaehler manifold of real dimension $2n$ with almost complex structure J and P a connected compact totally real submanifold of M with dimension n. Suppose that, for every $p \in P$ and every $N \in SN_pP$*

$$K(\gamma'_N(t), H_t) \geq (n-1)\lambda, \quad K(\gamma'_N(t), V_t) \geq (n-1)\lambda, \quad K_H(\gamma'_N(t)) \geq 4\lambda$$

and

$$k_{JN}(p) \geq 0, \quad H_{JN}(p) \geq 0.$$

Then

$$\frac{Vol(M)}{Vol(P)} \leq \frac{Vol(CP^n(\lambda))}{Vol(RP^n(\lambda))}, \tag{1.1}$$

where the holomorphic sectional curvature K_H of a $2n$-dimensional Kaehler manifold $(M, < , >, J)$ is the restriction of the sectional curvature of M to the holomorphic planes, and SN_pP denote the fibre of the unit normal bundle SNP at $p \in P$.

2. Preliminaries

Let M be a Riemannian manifold of dimension m, P a submanifold of M of dimension n $(< m)$. We shall denote by $NP(SNP)$ the (unit) normal bundle of P in M, and $N_PP(SN_PP)$ the fibre of $NP(SNP)$ at $p \in P$. Let $p \in P$ and $N \in SN_pP$. Let $\gamma_N(t)$ be geodesic of M satisfying $\gamma_N(0) = p$, $\gamma'_N(0) = N$. Let us denote by $M_t^\perp$ the orthogonal complement of $\gamma'_N(t)$ in $T_{\gamma_N(t)}M$. For $Y \in T_{\gamma_N(t)}M$, setting

$$R(t)Y := R(\gamma'_N(t), Y)\,\gamma'_N(t).$$

then $R(t)$ is a self-adjoint map of $M_t^\perp$.

From now on, M will denote a connected compact Kaehler manifold of real dimension $2n$ with almost complex structure J. P will denote a connected compact, totally real submanifold of M with dimension n.

Let $p \in P$, $N \in SN_pP$. Let $e_f(p, N) := inf\ \{t > 0 \mid \gamma_N(t)$ is a focal point of $P\}$. For every $t \in (0, e_f(p, N))$, there is a neighborhood U of $\gamma_N(t)$ and a neighborhood V of $p \in P$ such that $P(t) = U \cap \{m \in M \mid d(m, V) = t\}$ is a real hypersurface of M.

Let $S(t)$ be the Weingarten map of the hypersurface $P(t)$ associated to a unit normal vector field $N(t)$ defined on $P(t)$ as an extension of $\gamma'_N(t)$. Let ω be the Riemannian volume form of M, dp that of P and dN that of the unit sphere $S^{n-1}(1)$. We can define a smooth function $\theta_N(p, t)$ on $\{(p, N, t) \in SNP \times R \mid 0 < t < e_f(p, N)\}$ by

$$\omega(\gamma_N(t)) = \theta_N(x, t)t^{n-1} dN \wedge dP \wedge dt$$

Then the fundamental equations ([Gr1, 2]) for the real hypersurface $P(t)$ of M are given by

$$(2.1) \qquad S'(t) = S(t)^2 + R(t), \quad \text{where } S'(t) := \nabla_{\gamma'_N(t)} S(t),$$

$$(2.2) \qquad \frac{\theta'_N(p,t)}{\theta_N(x,t)} = -\left[trS(t) + \frac{n-1}{t}\right],$$

$$(2.3) \qquad \lim_{t \to 0} \theta_N(p, t) = 1.$$

3. The estimates for the volume element

For $p \in P$, $N \in SN_pP$, let $\{e_i\}_{1 \leq i \leq n}$ be an orthonormal basis of T_pP such that $e_n = JN$, and let $\{E_i(t)\}_{1 \leq i \leq n}$ be parallel vector fields along $\gamma_N(t)$ such that $E_i(0) = e_i$. Then $E_n(t) = J\gamma_N'(t)$. We shall consider the following orthogonal direct decomposition of

$$T_{\gamma_N(t)}M = H_t \oplus Span\ \{\gamma_N'(t),\ J\gamma_N'(t) = E_n(t)\} \oplus V_t,$$

where $H_t := Span\ \{E_1(t), \cdots, E_{n-1}(t)\}$.

Lemma 1. *Suppose that, for every $p \in P$, $N \in SN_pP$, $K_H(\gamma_N'(t)) \geq 4\lambda$, $K(\gamma_N'(t), H_t) \geq (n-1)\lambda$ and $K(\gamma_N'(t), V_t) \geq (n-1)\lambda$. Then*

$$\theta_N(p,t) \leq \mu_N(\lambda, p, t), \tag{3.1}$$

where $\mu_N(\lambda,p,t) = [\cos(\sqrt{\lambda}t) - \frac{H_{JN}(p)}{\sqrt{\lambda}} \sin(\sqrt{\lambda}t)]^{n-1}[\cos(2\sqrt{\lambda}t)$

$$-\frac{k_{JN}(p)}{2\sqrt{\lambda}} \sin(2\sqrt{\lambda}t)] \times \left[\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}t}\right]^{n-1}.$$

The equality in (3.1) is attained if and only if $< Le_i, e_i > = < Le_j, e_j > =: \beta$, $1 \leq i, j \leq n-1$, and with respect to $\{E_i\}_{1 \leq i \leq 2n-1}$, $S(t)$ and $R(t)$ have the matrix form

(3.2)

$$S(t) =$$

$$\begin{pmatrix}
-\frac{A'(\lambda,\beta,t)}{A(\lambda,\beta,t)} & & & & & & \\
 & n-1 & & & & & \\
 & \ddots & & & & 0 & \\
 & & -\frac{A'(\lambda,\beta,t)}{A(\lambda,\beta,t)} & & & & \\
 & & & -\frac{B'(\lambda,k_{JN},t)}{B(\lambda,k_{JN},t)} & & & \\
 & 0 & & & -\sqrt{\lambda}\cot(\sqrt{\lambda}t) & & \\
 & & & & & & n-1 \\
 & & & & & \ddots & \\
 & & & & & & -\sqrt{\lambda}\cot(\sqrt{\lambda}t)
\end{pmatrix}$$

$$R(t) = \begin{pmatrix} \lambda & & & & & & \\ & & n-1 & & & & \\ & \ddots & & & & 0 & \\ & & \lambda & & & & \\ & & & 4\lambda & & & \\ & & & & \lambda & & \\ & & & & & & n-1 \\ & 0 & & & & \ddots & \\ & & & & & & \lambda \end{pmatrix}$$

where $A(\lambda, \beta, t) = [\cos(\sqrt{\lambda}t) - (\frac{\beta}{\sqrt{\lambda}})\sin(\sqrt{\lambda}t)]$,

$$B(\lambda, k_{JN}, t) = [\cos(2\sqrt{\lambda}t) - (\frac{k_{JN}}{2\sqrt{\lambda}t})\sin(2\sqrt{\lambda}t)].$$

and $'$ denotes the derivative with respect to t.

Proof. We adopt an elementary proof due to [G] which is essentially the original proof of [Gr 2]. Let $\{e_1, \cdots, e_{n-1}, e_n = JN, e_{n+1} = Je_1, \cdots, e_{2n-1} = Je_{n-1}\}$ be an orthonormal basis of $N^{\perp}$ such that the basis $\{e_i\}_{1 \leq i \leq n}$ of T_pP diagonalizes the Weingarten map L_N of P associated to N, where $N^{\perp}$ is the orthogonal complement of the vector space spanned by N in T_pP. And let $\{E_i(t)\}_{1 \leq i \leq 2n-1}$ be parallel vector fields along $\gamma_N(t)$ such that $E_i(0) = e_i$.

Let us consider the functions

$$f_i(t) =< S(t)E_i(t), E_i(t) > . \tag{3.1}$$

Taking the derivative of both sides (3.1), using (2.1) and the Cauchy-Schwarz inequality, we get

$$f_i{}' \geq f_i{}^2 + < R(t)E_i, E_i > . \tag{3.2}$$

Then the hypotheses $K(\gamma_N'(t), H_t) \geq (n-1)\lambda$, $K_H(\gamma_N'(t)) \geq 4\lambda$ and $K(\gamma_N'(t), V_t) \geq (n-1)\lambda$ imply

$$\begin{gathered} \Big(\frac{1}{n-1}\sum_{i=1}^{n-1} f_i\Big)' \geq \Big(\frac{1}{n-1}\sum_{i=1}^{n-1} f_i^2\Big) + \lambda \geq \Big(\frac{1}{n-1}\sum_{i=1}^{n-1} f_i\Big)^2 + \lambda, \\ f_n' \geq f_n^2 + 4\lambda, \\ \Big(\frac{1}{n-1}\sum_{i=n+1}^{2n-1} f_i\Big)' \geq \Big(\frac{1}{n-1}\sum_{i=n+1}^{2n-1} f_i\Big)^2 + \lambda, \end{gathered} \tag{3.3}$$

with $f_i(0) =< Le_i, e_i >$, $i = 1, \cdots, n$, respectively. Then [Gr1] and the first and second inequalities gives

$$\frac{1}{n-1}\sum_{i=1}^{n-1} f_i(t) \ge \frac{\sqrt{\lambda}\sin(\sqrt{\lambda}t) + H_{JN}\cos(\sqrt{\lambda}t)}{\cos(\sqrt{\lambda}t) - (H_{JN}/\sqrt{\lambda})\sin(\sqrt{\lambda}t)} \tag{3.4}$$

$$f_n \ge \frac{2\sqrt{\lambda}\sin(2\sqrt{\lambda}t) + k_{JN}\cos(2\sqrt{\lambda}t)}{\cos(2\sqrt{\lambda}t) - (k_{JN}/2\sqrt{\lambda})\sin(2\sqrt{\lambda}t)} \tag{3.5}$$

respectively and the denominators of the right hand sides of these inequalities are positive from $t = 0$ to the first zero of each one. For $i = n+1, \cdots, 2n-1$ we have $f_i(0) = -\infty$. Then the third inequality and [Gr1] gives

$$\frac{1}{n-1}\sum_{i=n+1}^{2n-1} f_i \ge \frac{-\sqrt{\lambda}}{\tan(\sqrt{\lambda}t)} \tag{3.6}$$

Then, from (2.2)

$$\begin{aligned}\frac{d}{dt}\ln\theta_N(p,t) &= -[trS(t) + \frac{n-1}{t}] = -\sum_{i=1}^{2n-1} f_i(t) - \frac{n-1}{t}\\ &\le \frac{d}{dt}\ln\Bigg[(\cos(\sqrt{\lambda}t) - (\frac{H_{JN}}{\sqrt{\lambda t}})^{n-1})(\cos(2\sqrt{\lambda}t)\\ &\qquad -\frac{k_{JN}}{2\sqrt{\lambda}}\sin(2\sqrt{\lambda}t))(\frac{\sin(\sqrt{\lambda}t)}{t})^{n-1}\Bigg],\end{aligned}$$

from which (3.2) follows, taking account of (2.3).

If we have the equality in (3.1), then all inequalities in this proof must be equalities. The first equality in (3.3) implies $f_i(t) = f_j(t)$, $1 \le i, j \le n-1$, and $< Le_i, e_i >=< Le_j, e_j >= \beta$. Equalities in (3.1), (3.4) - (3.6) imply that $E_i(t)$ are eigenvectors of $S(t)$ with eigenvalues

$$f_i(t) = \begin{cases} \dfrac{\sqrt{\lambda}\sin(\sqrt{\lambda}t) + \beta\cos(\sqrt{\lambda}t)}{\cos(\sqrt{\lambda}t) - (\beta/\sqrt{\lambda})\sin(\sqrt{\lambda}t)} & \text{for} \quad 1 \le i \le n-1\\[2ex] \dfrac{2\sqrt{\lambda}\sin(2\sqrt{\lambda}t) + k_{JN}\cos(2\sqrt{\lambda}t)}{\cos(2\sqrt{\lambda}t) - (k_{JN}/2\sqrt{\lambda})\sin(2\sqrt{\lambda}t)} & \text{for} \quad i = n\\[2ex] -\sqrt{\lambda}\cot(\sqrt{\lambda}t) \quad \text{for} \quad n+1 \le i \le 2n-1 \end{cases} \tag{3.7}$$

The matrix form for $R(t)$ follows from (3.7) and the equality in (3.2). Q.E.D.

REMARK 2. Let $e_c(p,N) := sup\{t > 0 \mid d(p,\gamma_N(t)) = t\}$. Then $e_c(p,N) \leq e_f(p,N)$ (see, e.g., [He]). If $M = CP^n(\lambda)$ and $P = RP^n(\lambda)$, then $H_{JN} = 0$, $k_{JN} = 0$, $\theta_N(p,t) = \mu(t) := [\frac{\sin(2\sqrt{\lambda}t)}{2\sqrt{\lambda}t}]^{n-1}\cos(2\sqrt{\lambda}t)$ and $e_c(p,N) = \frac{\pi}{4\sqrt{\lambda}}$ for any $p \in RP^n(\lambda)$ and $N \in SN_pP$.

Given a function $q : X \times R^+ \to R$, where X is a given space, we denote by $z(q)$ the function which to every $x \in X$ associates the first zero of the function $t \to q(x,t)$.

LEMMA 3. *Suppose that $H_{JN} \geq 0$, $k_{JN} \geq 0$ on P. Then, for each fixed $p \in P$ and $N \in SN_pP$,*

$$\begin{aligned} \mu_N(\lambda,p,t) &\leq \mu(t),\ 0 \leq t \leq z(\mu_N(\lambda,p,t)), \\ z(\mu_N(\lambda,p,t)) &\leq \frac{\pi}{4\sqrt{\lambda}} \end{aligned}$$

Proof. Let $g(t) = \frac{\mu_N(\lambda,p,t)}{\mu(t)}$. Then $g'(t) \leq 0$ for $0 \leq t \leq z(\mu_N(\lambda,p,t))$ and $g(0) = 1$. The second assertion follows from the assumption. Q.E.D.

4. Proof of Theorem

It is well known ([Gr 1, 2]) that $z(\theta_N(p,t)) = e_f(p,N)$. Then we have from Lemma 1, 3 and Remark 2

$$\begin{aligned} Vol(M) &= \int_0^{e_c(p,N)} \int_P \int_{S^{n-1}(1)} \theta_N(p,t)t^{n-1}\,dN\,dP\,dt \\ &\leq \int_0^{\frac{\pi}{4\sqrt{\lambda}}} \int_P \int_{S^{n-1}(1)} \mu(t)t^{n-1}\,dN\,dp\,dt \\ &= \frac{Vol(P)}{Vol(RP^n(\lambda))} \int_0^{\frac{\pi}{4\sqrt{\lambda}}} \int_{RP^n(\lambda)} \int_{S^{n-1}(1)} t^{n-1}\mu(t)\,dN\,dp\,dt \\ &= \frac{Vol(P)}{Vol(RP^n(\lambda))} Vol(CP^n(\lambda)). \qquad \text{Q.E.D.} \end{aligned}$$

REMARK 4. (i) If the equality in (1.1) holds, then P is a totally geodesic submanifold of M. (ii) If $M = CP^n(\lambda)$, then the equality in (1.1) holds if and only if $P = RP^n(\lambda)$ (from [Ki]). If $P = RP^n(\lambda)$, then the equality in (1.1) holds if and only if $M = CP^n(\lambda)$ (from [Na]).

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