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SURFACES IN 4–DIMENSIONAL SPHERE

AKIRA YAMADA

1. Introduction

Let $\tilde{M} = (\tilde{M}, \tilde{J}, \langle\ \rangle)$ be an almost Hermitian manifold and M a submanifold of $\tilde{M}$. According to the behavior of the tangent bundle TM with respect to the action of $\tilde{J}$, we have two typical classes of submanifolds. One of them is the class of almost complex submanifolds and another is the class of totally real submanifolds. In 1990, B. Y. Chen [4],[5] introduced the concept of the class of slant submanifolds which involve the above two classes. He used the Wirtinger angle to measure the behavior of TM with respect to the action of $\tilde{J}$.

Let $J(M')$ be the metric twistor bundle over an even-dimensional oriented Riemannian manifold M' whose fiber $J_x(M')$ $(x \in M')$ consists of orthogonal complex structures compatible with the orientation of M'. We may define two kinds of natural almost Hermitian structures $(J_1, \langle\ ,\ \rangle_c)$ and $(J_2, \langle\ ,\ \rangle_c)$ on $J(M')$, where c is a positive real number and J_2 is never integrable. Many authors deal with these almost Hermitian structures in connection with the study of harmonic maps (cf. [1],[2],[6],[11],[12] and etc.). N. Ejiri [6] and other authors (cf. [2]) considered that the Calabi liftings $\Phi_+, \Phi_- : M \longrightarrow J(S^4)$ of an isometric immersion φ from an oriented Riemannian surface M into 4-dimensional unit sphere S^4, and obtained interesting results about the relationship between φ and $\Phi_\pm$, where Φ_+ (resp. Φ_-) denotes the positive Calabi lifting (resp. the negative Calabi lifting) of φ.

In this paper, we consider the positive Calabi lifting $\Phi = \Phi_+ : M \longrightarrow (J(S^4), J_1, \langle\ ,\ \rangle_c)$ (resp. $(J(S^4), J_2, \langle\ ,\ \rangle_c)$) of an isometric immersion φ from an oriented Riemannian surface M into S^4 by focusing our attention to the relationship between the Wirtinger angle of M in $J(S^4)$

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with respect to J_1 (resp. J_2) and the geometrical quantities with respect to φ, and prove the following Theorem.

THEOREM. *Let $\varphi : (M, g) \longrightarrow (S^4, \tilde{g})$ be an isometric immersion of an oriented Riemannian surface M into 4-dimensional unit sphere S^4, $\Phi : M \longrightarrow (J(S^4), J_1, \langle\ ,\ \rangle_c)$ (resp. $(J(S^4), J_2, \langle\ ,\ \rangle_c)$) the positive Calabi lifting of φ and α_1 (resp. α_2) the Wirtinger angle of M in $J(S^4)$ with respect to J_1 (resp. J_2). Then we have the following equalities,*

(i) $4c^2\|H\|^2 \cos^2 \alpha_1 = \{1 - c^2(-1 + \kappa + \kappa_N)\}^2 \sin^2 \alpha_1 + 4c^2(-1 + \kappa + \kappa_N)$,

(ii) $4c^2\|H\|^2 \cos^2 \alpha_2 = \{1 - c^2(-1 + \kappa + \kappa_N)\}^2 \sin^2 \alpha_2$,

where H is the mean curvature vector of M with respect to φ, κ is the Gaussian curvature of M and κ_N is the normal Gaussian curvature of M with respect to φ.

By using the equalities in the above Theorem, we may obtain another proof of the result of M. F. Atiyah-H. B. Lawson (cf. [6],[7]).

COROLLARY 1. *Let $\varphi : (M, g) \longrightarrow (S^4, \tilde{g})$ be an isometric immersion and $\Phi : M \longrightarrow (J(S^4), J_1, \langle\ ,\ \rangle_c)$ (or $(J(S^4), J_2, \langle\ ,\ \rangle_c)$) the positive Calabi lifting of φ. Then, we have the following.*

(i) *We suppose that φ is minimal. Then, Φ is holomorphic with respect to J_1 if and only if φ is super-minimal.*

(ii) *Φ is pseudo-holomorphic with respect to J_2 if and only if φ is minimal.*

Since A. Nijenhuis and W. B. Woolf showed that every almost complex manifolds has a (local) holomorphic curve passing through any point with any complex tangent vector (Theorem III of [9]), we may have a (local) J_2-holomorphic curve in $J(S^4)$ passing through any point with any complex tangent vector. By Corollary 1 (ii), we may construct many minimal surfaces in S^4 locally by projecting its J_2-holomorphic curves in $J(S^4)$ onto S^4 via the bundle projection $\pi_1 : J(S^4) \longrightarrow S^4$.

From the above Theorem, we may also obtain the following.

COROLLARY 2. *Let $\varphi : (M, g) \longrightarrow (S^4, \tilde{g})$ be an isometric immersion and $\Phi : M \longrightarrow (J(S^4), J_1, \langle\ ,\ \rangle_c)$ (or $(J(S^1), J_2, \langle\ ,\ \rangle_c)$) the positive Calabi lifting of φ. Then, we have the following.*

(i) *Φ is totally real with respect to $(J_1, \langle\ ,\ \rangle_c)$ if and only if $\kappa + \kappa_N = 1 - \frac{1}{c^2}$.*

(ii) Φ *is totally real with respect to* $(J_2, \langle\ ,\ \rangle_c)$ *if and only if* $\kappa + \kappa_N = 1 + \frac{1}{c^2}$.

In the case of $c = 1$, Corollary 2 (i) gives the result of N. Ejiri [6].

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2. Preliminaries

Let $\Phi : M \longrightarrow \tilde{M}$ be an immersion of a C^∞-manifold M into a $2n$-dimensional almost Hermitian manifold $\tilde{M} = (\tilde{M}, \tilde{J}, \langle\ ,\ \rangle)$. We endow M with the induced metric via Φ. We identify the tangent space T_xM at a point $x \in M$ and its image $(\Phi_*)_x T_xM$ of Φ_*, and denote them by T_xM in the case there is no danger of confusion. For any nonzero vector $X \in T_xM$, the angle $\theta_x(X)$ between $\tilde{J}X$ and the tangent space T_xM at $x \in M$ is called the *Wirtinger angle* of X.

$$\theta_x(X) := \angle(\tilde{J}X, T_xM), \quad 0 \leq \theta_x(X) \leq \frac{\pi}{2} \tag{2.1}$$

In general, the Wirtinger angle $\theta_x(X)$ depends on the choice of the point $x \in M$ and the vector $X \in T_xM$. If the Wirtinger angle $\theta_x(X)$ is constant for any point $x \in M$ and vector $X \in T_xM$, the immersion Φ is called the *slant immersion.* Almost complex (or holomorphic) immersion (resp. totally real immersion) is a slant immersion with $\theta = 0$ (resp. $\theta = \pi/2$).

It is easily seen that, if $\dim M = 2$, then the Wirtinger angle depends only on the choice of the point $x \in M$; i.e. $\theta_x(X) = \theta(x)$ and $\theta(x)$ is given by

$$\cos\theta(x) = |\ \langle \tilde{J}X_1, X_2\rangle\ |, \tag{2.2}$$

where $\{X_1, X_2\}$ is an orthonormal basis of T_xM.

We shall now review some fundamental facts on almost Hermitian structures on the metric twistor bundle $J(S^4)$ over S^4 (in detail, see [12]). We adopt the same notational convention as used in [12]. Let $S^4 = (S^4, \tilde{g})$ be 4-dimensional unit sphere with the fixed orientation and $\pi : F(S^4) \longrightarrow S^4$ the oriented orthonormal frame bundle. We denote by θ and ω the canonical form and the connection form on $F(S^4)$ with respect to the Riemannian connection $\tilde{\nabla}$ of $\tilde{g}$. The structure group of the principal fiber bundle $F(S^4)$ is the special orthogonal group $SO(4)$ of degree 4. We denote by $\mathfrak{so}(4)$ the Lie algebra of $SO(4)$. Let $\mathbb{R}^4$ be the 4-dimensional Euclidean space with the canonical inner product $\xi \cdot \eta$ for $\xi, \eta \in \mathbb{R}^4$, and J_0 the linear endomorphism of $\mathbb{R}^4$ given by

$$J_0 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.3}$$

with respect to the canonical orthonormal basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_4\}$ of $\mathbb{R}^4$. We denote by A^* (resp. $B(\xi)$) the fundamental vector field (resp. the basic vector field) corresponding to $A \in \mathfrak{so}(4)$ (resp. $\xi \in \mathbb{R}^4$). For each $u \in F(S^4)$, we define a linear endomorphism $j(u)$ on $T_{\pi(u)}S^4$ by

$$j(u) := u \circ J_0 \circ u^{-1}. \tag{2.4}$$

Then, by (2.3) and (2.4), we see immediately that $j(u)$ is an orthogonal almost complex structure at $\pi(u)$ compatible with the orientation of S^4. The linear endomorphism $j(u)$ is called a *metric twistor* at $\pi(u)$. For each point $x \in S^4$, we put $J_x(S^4) := \{j(u) | \pi(u) = x\}$. Then we may easily see that $J_x(S^4)$ is diffeomorphic to $S^2 = SO(4)/U(2)$ ($U(2) = \{a \in SO(4) | aJ_0 = J_0 a\}$ (unitary group of degree 2)). We put $J(S^4) := \bigcup_{x \in S^4} J_x(S^4)$, then it is known that $j : F(S^4) \longrightarrow J(S^4)$ is a principal fiber bundle with the structure group $U(2)$ and hence $J(S^4)$ is the associated fiber bundle of $F(S^4)$ with the standard fiber S^2. The fiber bundle $\pi_1 : J(S^4) \longrightarrow S^4$ is called the *metric twistor bundle* over S^4. It is easily seen that the total space $J(S^4)$ is diffeomorphic to $\mathbb{C}P^3$. Then we have the following commutative diagram :

$$\begin{array}{ccc} J(S^4) & \xleftarrow{\ j\ } & F(S^4) \\ {\scriptstyle \pi_1}\downarrow & & \| \\ S^4 & \xleftarrow[\ \pi\ ]{} & F(S^4). \end{array} \tag{2.5}$$

Next, we consider the standard fiber $S^2 = SO(4)/U(2)$. Let σ be the involutive automorphism of $SO(4)$ defined by

$$\sigma(a) := -J_0 a J_0, \quad \text{for } a \in SO(4). \tag{2.6}$$

Then, by (2.6), we see immediately that $SO(4)^\sigma = \{a \in SO(4) | \sigma(a) = a\} = U(2)$. Furthermore, we have the corresponding Cartan decomposition of $\mathfrak{so}(4)$:

$$\mathfrak{so}(4) = \mathfrak{u}(2) \oplus \mathfrak{m}, \tag{2.7}$$

where $\mathfrak{u}(2)$ denotes the Lie algebra of $U(2)$. Concretely,

$$\begin{aligned}
A =& \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} \in \mathfrak{so}(4), \\
B =& \frac{1}{2} \begin{pmatrix} 0 & a+f & 2b & c+d \\ -(a+f) & 0 & c+d & 2e \\ -2b & -(c+d) & 0 & a+f \\ -(c+d) & -2e & -(a+f) & 0 \end{pmatrix} \in \mathfrak{u}(2), \\
C =& \frac{1}{2} \begin{pmatrix} 0 & a-f & 0 & c-d \\ -(a-f) & 0 & -(c-d) & 0 \\ 0 & c-d & 0 & -(a-f) \\ -(c-d) & 0 & a-f & 0 \end{pmatrix} \in \mathfrak{m}, \\
A =& B + C,
\end{aligned} \tag{2.8}$$

where $a, b, c, d, e, f \in \mathbb{R}$. By (2.8), we see that the elements of $\mathfrak{m}$ can be represented by (1,2)- and (1,4)-components, so we denote the elements of $\mathfrak{m}$ as following,

$$[a : b] := \begin{pmatrix} 0 & a & 0 & b \\ -a & 0 & -b & 0 \\ 0 & b & 0 & -a \\ -b & 0 & a & 0 \end{pmatrix} \in \mathfrak{m}. \tag{2.9}$$

We see that $J_0[a : b] = [-b : a] \in \mathfrak{m}$ and $Ad(a)J_0 = J_0 Ad(a)$ on $\mathfrak{m}$ for all $a \in U(2)$. Thus, J_0 gives rise to an $SO(4)$-invariant almost complex structure on S^2. We define an inner product $(\ ,\ )$ on $\mathfrak{so}(4)$ by

$$(A, B) = -\mathrm{trace}(AB), \tag{2.10}$$

for $A, B \in \mathfrak{so}(4)$. Then we may easily see that the inner product $(\ ,\ )$ gives rise to a biinvariant Riemannian metric on $SO(4)$ (and hence an $SO(4)$-invariant Riemannian metric on S^2) and furthermore $(J_0, (\ ,\ ))$ is an almost Hermitian structure on S^2. Corresponding to the decomposition (2.7), we may write

$$\omega = \omega_1 + \omega_2, \tag{2.11}$$

where ω_1 (resp. ω_2) denotes $\mathfrak{u}(2)$-component (resp. $\mathfrak{m}$-component) of ω. Then, by taking account of (2.5),(2.7) and (2.11), we see that there exists a linear isomorphism $\lambda(u) : T_{j(u)}J(S^4) \longrightarrow \mathfrak{m} \oplus \mathbb{R}^4$ satisfying the following two conditions:

$$\lambda(ua) = \left(Ad(a^{-1}) \oplus a^{-1}\right) \lambda(u), \quad \text{for } a \in U(2), \tag{2.12}$$

and the diagram

$$\begin{array}{ccc} T_uF(S^4) & \xrightarrow{(j_*)_u} & T_{j(u)}J(S^4) \\ \| & & \downarrow \lambda(u) \\ T_uF(S^4) & \xrightarrow[(\omega_2+\theta)_u]{} & \mathfrak{m} \oplus \mathbb{R}^4 \end{array} \tag{2.13}$$

is commutative for any $u \in F(S^4)$. We put

$$\begin{aligned} H(j(u)) &:= \lambda(u)^{-1}(\mathbb{R}^4) \\ V(j(u)) &:= \lambda(u)^{-1}(\mathfrak{m}), \end{aligned} \tag{2.14}$$

for each $u \in F(S^4)$. Then H and V give rise to differentiable distributions on $J(S^4)$ which are called the ***horizontal distribution*** and the ***vertical distribution*** on $J(S^4)$, respectively.

We define (1,1)-type tensor fields J_1', J_2' on $F(S^4)$ by

$$\begin{aligned} &J_1'A^* := 0, \quad J_2'A^* := 0, \quad \text{for } A \in \mathfrak{u}(2) \\ &J_1'A^* := (J_0A)^*, \quad J_2'A^* := -(J_0A)^*, \quad \text{for } A \in \mathfrak{m}, \\ &J_1'B(\xi) := B(J_0\xi), \quad J_2'B(\xi) := B(J_0\xi), \quad \text{for } \xi \in \mathbb{R}^4. \end{aligned} \tag{2.15}$$

Taking account of (2.13), we may define almost complex structures J_1, J_2 on $J(S^4)$ by

$$
(2.16)\qquad
\begin{aligned}
&(J_1)_{j(u)}((j_*)_u A_u^*) := \lambda(u)^{-1}(J_0 A), \quad \text{for } A \in \mathfrak{m},\\
&(J_1)_{j(u)}((j_*)_u B(\xi)_u) := \lambda(u)^{-1}(J_0 \xi), \quad \text{for } \xi \in \mathbb{R}^4,\\
&(J_2)_{j(u)}((j_*)_u A_u^*) := -\lambda(u)^{-1}(J_0 A), \quad \text{for } A \in \mathfrak{m},\\
&(J_2)_{j(u)}((j_*)_u B(\xi)_u) := \lambda(u)^{-1}(J_0 \xi), \quad \text{for } \xi \in \mathbb{R}^4,
\end{aligned}
$$

at $j(u) \in J(S^4)$. By (2.13),(2.15) and (2.16), we get immediately

$$
(2.17)\qquad J_1 \circ j_* = j_* \circ J_1', \quad J_2 \circ j_* = j_* \circ J_2'.
$$

It is known that J_2 is never integrable. On the other hand, J_1 is integrable by self-duality of S^4 (see [1],[11]).

Next, we give a Riemannian metric $\langle\ ,\ \rangle_c'$ (c is a positive real number) on $F(S^4)$ by

$$
(2.18)\qquad
\begin{aligned}
&\langle A^*, B^* \rangle_c' := c^2(A, B),\\
&\langle A^*, B(\xi) \rangle_c' := 0,\\
&\langle B(\xi), B(\eta) \rangle_c' := \xi \cdot \eta,
\end{aligned}
$$

for $A, B \in \mathfrak{so}(4)$, $\xi, \eta \in \mathbb{R}^4$. Furthermore, by taking account of (2.10), we may define a Riemannian metric $\langle\ ,\ \rangle_c$ on $J(S^4)$ by

$$
(2.19)\qquad
\begin{aligned}
&\langle j_* A^*, j_* B^* \rangle_c := c^2(A, B),\\
&\langle j_* A^*, j_* B(\xi) \rangle_c := 0,\\
&\langle j_* B(\xi), j_* B(\eta) \rangle_c := \xi \cdot \eta,
\end{aligned}
$$

for $A, B \in \mathfrak{m}$, $\xi, \eta \in \mathbb{R}^4$. Then, by (2.18) and (2.19), we see that $j : (F(S^4), \langle\ ,\ \rangle_c') \longrightarrow (J(S^4), \langle\ ,\ \rangle_c)$ is a Riemannian submersion. Also, by (2.16) and (2.19), we have that $(J_1, \langle\ ,\ \rangle_c)$ and $(J_2, \langle\ ,\ \rangle_c)$ are almost Hermitian structures on $J(S^4)$. It is known that $(J(S^4), J_1, \langle\ ,\ \rangle_1)$ is a Kählerian manifold and $(J(S^4), J_2, \langle\ ,\ \rangle_{1/\sqrt{2}})$ is a nearly Kählerian manifold ([12]).

3. Calabi liftings

Let $M = (M, g)$ be an oriented Riemannian surface and $\varphi : (M, g) \longrightarrow (S^4, \tilde{g})$ an isometric immersion. We may see that $M = (M, J, g)$ is a Hermitian manifold with the natural complex structure J. For any point $x \in M$, we have the orthogonal decomposition $T_x S^4 = T_x M \oplus T_x^{\perp} M$. For each point $x \in M$, we take the oriented orthonormal frame $u = (x; e_1, e_2, e_3, e_4) \in F(S^4)$ of S^4 such that

$$e_1, e_3 := Je_1 \in T_x M, \quad e_2, e_4 \in T_x^{\perp} M. \tag{3.1}$$

Then $T^{\perp} M$ has the natural orientation determined by the orientations of M and S^4. So we may define the almost complex structure $J^{\perp}$ of $T^{\perp} M$ by

$$J^{\perp} e_2 := e_4, \quad J^{\perp} e_4 := -e_2. \tag{3.2}$$

We remark that the definition of $J^{\perp}$ is well-defined. For each point $x \in M$, we define the metric twistor j_x by

$$j_x := J_x \oplus J_x^{\perp} \in J(S^4). \tag{3.3}$$

Then, j_x has following relation to $j(u)$,

$$j_x = j(u) = u \circ J_0 \circ u^{-1}, \quad \text{where } \pi(u) = x.$$

We define the map $\Phi : M \longrightarrow J(S^4)$ by

$$\Phi(x) := j_x. \tag{3.4}$$

We see that Φ is well-defined. This map Φ is called the ***positive Calabi lifting*** of φ. Choosing the reverse orientation of S^4, we have another map of M into $J(S^4)$ which is called the ***negative Calabi lifting*** of φ.

$$\begin{array}{ccccc} M & \xrightarrow{\ \Phi\ } & J(S^4) & \xleftarrow{\ j\ } & F(S^4) \\ \| & & \downarrow{\scriptstyle \pi_1} & & \| \\ M & \xrightarrow[\ \varphi\ ]{} & S^4 & \xleftarrow[\ \pi\ ]{} & F(S^4) \end{array} \tag{3.5}$$

In the rest of this section, we prepare some equalities and Lemmas for proofs of Theorem and Corollaries. Let $\nabla, \tilde{\nabla}$ be the Riemannian connections of M, S^4 with respect to $g, \tilde{g}$, respectively, and σ the second fundamental form of M with respect to φ, A the shape operator of M with respect to φ, $\nabla^{\perp}$ the normal connection of $T^{\perp}M$ with respect to φ, H the mean curvature vector of M with respect to φ, κ the Gaussian curvature of M and κ_N the normal Gaussian curvature of M with respect to φ. For the point $x \in M$ such that $\sigma \neq 0$, we consider the map from T_xM into $T_x^{\perp}M$ given by

$$X \in T_xM(\|X\| = 1) \longmapsto \sigma(X, X) \in T_x^{\perp}M. \tag{3.6}$$

We define the oriented orthonormal frame $u = (x; e_1, e_2, e_3, e_4) \in F(S^4)$ by

$$\begin{aligned} &\|\sigma(e_1, e_1)\| := \max_{\|X\|=1} \|\sigma(X, X)\|, \text{ where } X \in T_xM, \\ &e_2 := \frac{\sigma(e_1, e_1)}{\|\sigma(e_1, e_1)\|}, \quad e_3 := Je_1, \quad e_4 := J^{\perp}e_2. \end{aligned} \tag{3.7}$$

This frame is called an *E-frame.* We consider the geodesic γ in M passing through $x \in M$ with the initial vector $\dot{\gamma}(0) = X \in T_xM$:

$$\gamma(t) := \exp_x(tX). \tag{3.8}$$

Then, we get a ∇ (resp. $\nabla^{\perp}$)-parallel vector field $e_1(t)$ (resp. $e_2(t)$) such that $e_1(0) = e_1$ (resp. $e_2(0) = e_2$) by the parallel translation along γ with respect to ∇ (resp. $\nabla^{\perp}$). Thus, we get a $\nabla, \nabla^{\perp}$-parallel frame field $u(t)$ along γ:

(3.9)

$$u(t) = \big(\gamma(t); e_1(t), e_2(t), e_3(t) = Je_1(t), e_4(t) = J^{\perp}e_2(t)\big) \in F(S^4).$$

From now on, we use the range of indices: $i, j = 1, 3$ and $\alpha = 2, 4$. With respect to this local frame field, we obtain

$$\begin{aligned} (\,\tilde{g}(\sigma(e_i, e_j), e_2)\,) &= \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \\ (\,\tilde{g}(\sigma(e_i, e_j), e_4)\,) &= \begin{pmatrix} 0 & \nu \\ \nu & \delta \end{pmatrix}, \end{aligned} \tag{3.10}$$

where λ, μ, ν and δ are locally smooth functions. We remark that, at a geodesic point i.e. $\sigma = 0$, we may consider $\lambda = \mu = \nu = \delta = 0$. By the definition of H, the equation of Gauss and the equation of Ricci, we easily obtain the following equalities:

$$\|H\|^2 = \frac{1}{4}\{(\lambda + \mu)^2 + \delta^2\}, \tag{3.11}$$

$$\kappa = 1 + \lambda\mu - \nu^2, \tag{3.12}$$

$$\kappa_N = \nu(\lambda - \mu). \tag{3.13}$$

We consider the image

$$E_x := \{\sigma(X, X) | X \in T_x M, \|X\| = 1\} \subset T_x^{\perp} M \tag{3.14}$$

of the map (3.6) which is called the *ellipse of curvature*.

LEMMA 1. *Ellipse of curvature E_x at $x \in M$ is a circle if and only if*

$$\nu = \pm\frac{\lambda - \mu}{2}, \quad \delta = 0.$$

In particular, the map (3.6) preserves or reverses the orientation according as $\nu = (\lambda - \mu)/2$ or $\nu = -(\lambda - \mu)/2$.

If the ellipse of curvature Ex is a circle and the map (3.6) preserves (resp. reverses) the orientation, E_x is called the *positive* (resp. *negative*) *circle*. In particular, a minimal immersion of M into S^4 is called *super-minimal* if and only if the ellipse of curvature is a positive circle. Taking account of (3.10) and Lemma 1, we have the following.

LEMMA 2. *φ is super-minimal if and only if the second fundamental form σ of M with respect to φ is of the following forms with respect to the local frame field (3.9),*

$$(\tilde{g}(\sigma(e_i, e_j), e_2)) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix},$$

$$(\tilde{g}(\sigma(e_i, e_j), e_4)) = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix},$$

where λ is locally smooth function.

Next, We shall calculate the differential map Φ_* of the positive Calabi lifting Φ. We denote by $\bar{X}$ the tangent vector of $u(t)$ at $u = u(0)$,

$$\bar{X} := \left.\frac{d}{dt}\right|_{t=0} u(t) \in T_u F(S^4). \tag{3.15}$$

Then, by (2.13),(3.4),(3.5),(3.8),(3.9) and (3.15), we have following series of equalities:

$$\pi \circ u(t) = \gamma(t), \tag{3.16}$$
$$(\pi_*)_u \bar{X} = \dot{\gamma}(0) = X, \tag{3.17}$$
$$(\pi_{1*} \circ \Phi_*)_x X = (\varphi_*)_x X, \tag{3.18}$$
$$\Phi(\gamma(t)) = u(t) \circ J_0 \circ u(t)^{-1} = j(u(t)), \tag{3.19}$$
$$(\Phi_*)_x X = (j_*)_u \bar{X}, \tag{3.20}$$
$$\lambda(u)(\Phi_*)_x X = (\omega_2)_u(\bar{X}) + \theta_u(\bar{X}). \tag{3.21}$$

Now we put

$$\theta_u(\bar{X}) = u^{-1}(\pi_*)_u \bar{X} = u^{-1} X =: \xi. \tag{3.22}$$

We shall calculate $\omega(\bar{X})$. With respect to the frame field (3.9), we have

$$\tilde{\nabla}_X e_i = \sigma(X, e_i) = \tilde{g}(\sigma(X, e_i), e_2)e_2 + \tilde{g}(\sigma(X, e_i), e_4)e_4,$$
$$\tilde{\nabla}_X e_\alpha = -A_{e_\alpha} X = -\tilde{g}(\sigma(X, e_1), e_\alpha)e_1 - \tilde{g}(\sigma(X, e_3), e_\alpha)e_3.$$

If we express

$$(\tilde{\nabla}_X e_1, \tilde{\nabla}_X e_2, \tilde{\nabla}_X e_3, \tilde{\nabla}_X e_4) = (e_1, e_2, e_3, e_4)\omega(\bar{X}),$$

then we have

$$\omega(\bar{X}) = \begin{pmatrix} 0 & -\tilde{g}(\sigma(X, e_1), e_2) & 0 & -\tilde{g}(\sigma(X, e_1), e_4) \\ \tilde{g}(\sigma(X, e_1), e_2) & 0 & \tilde{g}(\sigma(X, e_3), e_2) & 0 \\ 0 & -\tilde{g}(\sigma(X, e_3), e_2) & 0 & -\tilde{g}(\sigma(X, e_3), e_4) \\ \tilde{g}(\sigma(X, e_1), e_4) & 0 & \tilde{g}(\sigma(X, e_3), e_4) & 0 \end{pmatrix}$$

Thus, by (2.8),(2.9) and (2.11),

$$\omega_2(X) = \frac{1}{2}[a(X) : b(X)], \tag{3.23}$$

where

$$\begin{aligned} a(X) &:= \tilde{g}(\sigma(X, e_3), e_4) - \tilde{g}(\sigma(X, e_1), e_2), \\ b(X) &:= -\tilde{g}(\sigma(X, e_1), e_4) - \tilde{g}(\sigma(X, e_3), e_2). \end{aligned} \tag{3.24}$$

By (2.16) and (3.20)~(3.23), we have the following equalities:

$$(\Phi_*)_x X = (j_*)_u B(\xi)_u + \frac{1}{2}(j_*)_u [a(X) : b(X)]^*_u, \tag{3.25}$$

$$J_1(\Phi_*)_x X = (j_*)_u B(J_0\xi)_u + \frac{1}{2}(j_*)_u [-b(X) : a(X)]^*_u, \tag{3.26}$$

$$J_2(\Phi_*)_x X = (j_*)_u B(J_0\xi)_u + \frac{1}{2}(j_*)_u [b(X) : -a(X)]^*_u. \tag{3.27}$$

4. Proofs

In this section, we prove Theorem and Corollary 1,2.

Proof of Theorem. In general, g does not coincide with the induced metric via Φ. So we first seek an orthonormal basis $\{X_1, X_2\}$ of $(\Phi_*)_x T_x M$ with respect to $\langle\ ,\ \rangle_c$. By (3.25), we have

$$\begin{aligned} (\Phi_*)_x e_1 &= (j_*)_u B(\mathbf{e}_1)_u + \frac{1}{2}(j_*)_u [a(e_1) : b(e_1)]^*_u, \\ (\Phi_*)_x e_3 &= (j_*)_u B(\mathbf{e}_3)_u + \frac{1}{2}(j_*)_u [a(e_3) : b(e_3)]^*_u. \end{aligned}$$

To get the length of $(\Phi_*)_x e_1$ and $(\Phi_*)_x e_3$, we calculate the followings:

$$\begin{aligned} ([a(e_1) : b(e_1)], [a(e_1) : b(e_1)]) &= 4\{a(e_1)^2 + b(e_1)^2\} \\ &= 4(\lambda - \nu)^2, \end{aligned} \tag{4.1}$$

(4.2)
$$([a(e_1):b(e_1)],[a(e_3):b(e_3)]) = 4\{a(e_1)a(e_3)+b(e_1)b(e_3)\} = 4\delta(\nu-\lambda),$$

(4.3)
$$([a(e_3):b(e_3)],[a(e_3):b(e_3)]) = 4\{a(e_3)^2+b(e_3)^2\} = 4\{\delta^2+(\mu+\nu)^2\}.$$

By (2.19) and (4.1)~(4.3), we get

$$\langle(\Phi_*)_x e_1,(\Phi_*)_x e_1\rangle_c = 1+c^2(\lambda-\nu)^2, \tag{4.4}$$

$$\langle(\Phi_*)_x e_1,(\Phi_*)_x e_3\rangle_c = c^2\delta(\nu-\lambda), \tag{4.5}$$

$$\langle(\Phi_*)_x e_3,(\Phi_*)_x e_3\rangle_c = 1+c^2\{\delta^2+(\mu+\nu)^2\}. \tag{4.6}$$

By applying the Gram-Schmidt orthonormalization to $(\Phi_*)_x e_1$ and $(\Phi_*)_x e_3$, we make an orthonormal basis $\{X_1, X_2\}$ of $(\Phi_*)_x T_x M$ with respect to $\langle\ ,\ \rangle_c$:

(4.7)
$$X_1 = \frac{1}{\sqrt{1+c^2(\lambda-\nu)^2}}\left\{(j_*)_u B(\mathbf{e}_1)_u + \frac{1}{2}(j_*)_u[a(e_1):b(e_1)]^*_u\right\},$$

(4.8)
$$\begin{aligned}X_2 =& \frac{1}{L}\Bigg[(j_*)_u B(\mathbf{e}_3)_u + \frac{c^2\delta(\lambda-\nu)}{1+c^2(\lambda-\nu)^2}(j_*)_u B(\mathbf{e}_1)_u \\ &+ \frac{1}{2}\left\{(j_*)_u[a(e_3):b(e_3)]^*_u + \frac{c^2\delta(\lambda-\nu)}{1+c^2(\lambda-\nu)^2}(j_*)_u[a(e_1):b(e_1)]^*_u\right\}\Bigg],\end{aligned}$$

where

$$L := \sqrt{\frac{1+c^2(\lambda-\nu)^2+c^2\delta^2+c^2(\mu+\nu)^2\{1+c^2(\lambda-\nu)^2\}}{1+c^2(\lambda-\nu)^2}}. \tag{4.9}$$

By (2.2),(3.26),(3.27) and (4.1)~(4.8), we have

$$\cos\alpha_1 = \frac{|1+c^2(\lambda-\nu)(\mu+\nu)|}{L\sqrt{1+c^2(\lambda-\nu)^2}}, \tag{4.10}$$

$$\cos\alpha_2 = \frac{|1-c^2(\lambda-\nu)(\mu+\nu)|}{L\sqrt{1+c^2(\lambda-\nu)^2}}. \tag{4.11}$$

For the sake of simplicity, we put $A := \lambda - \nu$ and $B := \mu + \nu$. Then $A + B = \lambda + \mu$. By (3.11)~(3.13), we get

$$\|H\|^2 = \frac{1}{4}\{(\lambda + \mu)^2 + \delta^2\} = \frac{1}{4}\{(A + B)^2 + \delta^2\},$$
$$AB = (\lambda - \nu)(\mu + \nu) = -1 + \kappa + \kappa_N.$$

We square the both sides of (4.10) and express by A, B and δ.

$$\cos^2 \alpha_1 = \frac{(1 + c^2 AB)^2}{1 + c^2 A^2 + c^2\delta^2 + c^2 B^2(1 + c^2 A^2)}$$
$$4c^2\|H\|^2 \cos^2 \alpha_1 = (1 - c^2 AB)^2 \sin^2 \alpha_1 + 4c^2 AB$$

Thus, we obtain

$$4c^2\|H\|^2 \cos^2 \alpha_1 = \{1 - c^2(-1 + \kappa + \kappa_N)\}^2 \sin^2 \alpha_1 - 4c^2(-1 + \kappa + \kappa_N).$$

We square the both sides of (4.11) and express by A, B and δ.

$$\cos^2 \alpha_2 = \frac{(1 - c^2 AB)^2}{1 + c^2 A^2 + c^2\delta^2 + c^2 B^2(1 + c^2 A^2)}$$
$$4c^2\|H\|^2 \cos^2 \alpha_2 = (1 - c^2 AB)^2 \sin^2 \alpha_2$$

Thus, we obtain

$$4c^2\|H\|^2 \cos^2 \alpha_2 = \{1 - c^2(-1 + \kappa + \kappa_N)\}^2 \sin^2 \alpha_2.$$

This completes the proof of Theorem. □

REMARK. By (4.4)~(4.6) and Lemma 2, we easily see that g coincides with the induced metric via Φ if and only if φ is super-minimal.

Proof of Corollary 1. (i) We suppose that Φ is holomorphic with respect to J_1, that is, $\alpha_1 = 0$. By Theorem (i), we have

$$-1 + \kappa + \kappa_N = 0, \quad \text{i.e.} \quad (\lambda - \nu)(\mu + \nu) = 0.$$

By (3.11), we have $\lambda + \mu = 0$ and $\delta = 0$. Therefore we get $\mu = -\lambda, \nu = \lambda$ and $\delta = 0$. By Lemma 2, φ is super-minimal.

Conversely, we suppose that φ is super-minimal. By Lemma 2, we have $\mu = -\lambda, \nu = \lambda$ and $\delta = 0$. Thus we have

$$-1 + \kappa + \kappa_N = 0.$$

Then, by Theorem (i), we get $\alpha_1 = 0$. Therefore Φ is holomorphic with respect to J_1.

(ii) We suppose that Φ is pseudo-holomorphic with respect to J_2, that is, $\alpha_2 = 0$. By Theorem (ii), we get $H = 0$. Hence φ is minimal.

Conversely, we suppose that φ is minimal. By Theorem (ii), we have

$$1 - c^2(-1 + \kappa + \kappa_N) = 0 \quad \text{or} \quad \alpha_2 = 0.$$

On the other hand, by (3.11), $\lambda + \mu = 0$ and $\delta = 0$, so we have

$$1 - c^2(-1 + \kappa + \kappa_N) = 1 + c^2(\lambda - \nu)^2 \neq 0.$$

Therefore, $\alpha_2 = 0$ and so Φ is pseudo-holomorphic with respect to J_2. □

Proof of Corollary 2. (i) We suppose that Φ is totally real with respect to $(J_1, \langle\ ,\ \rangle_c)$, that is, $\alpha_1 = \pi/2$. By Theorem (i), we get $\kappa + \kappa_N = 1 - 1/c^2$.

Conversely, we suppose that $\kappa + \kappa_N = 1 - 1/c^2$. By Theorem (i), we have

$$(1 + c^2\|H\|^2)\cos^2\alpha_1 = 0$$

Since $1 + c^2\|H\|^2 \neq 0$, we get $\alpha_1 = \pi/2$ and so Φ is totally real with respect to $(J_1, \langle\ ,\ \rangle_c)$.

(ii) We suppose that Φ is totally real with respect to $(J_2, \langle\ ,\ \rangle_c)$, that is, $\alpha_2 = \pi/2$. By Theorem (ii), we get $\kappa + \kappa_N = 1 + 1/c^2$.

Conversely, we suppose that $\kappa + \kappa_N = 1 + 1/c^2$. By Theorem (ii), we have

$$\|H\|^2 \cos^2\alpha_2 = 0.$$

Thus $H = 0$ or $\alpha_2 = \pi/2$. On the other hand, by $\kappa + \kappa_N = 1 + 1/c^2$, we have

$$(\lambda - \nu)(\mu + \nu) = \frac{1}{c^2},$$

$$\lambda = \frac{1}{c^2(\mu + \nu)} + \nu,$$

$$\lambda + \mu = \frac{1 + c^2(\mu + \nu)^2}{c^2(\mu + \nu)} \neq 0.$$

Hence by (3.11), we have $H \neq 0$. Thus we get $\alpha_2 = \pi/2$ and so Φ is totally real with respect to $(J_2, \langle\ ,\ \rangle_c)$. □

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Department of Mathematical Science
Graduate school of science and technology
Niigata University
Niigata 950-21, Japan