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ON TRANSFORMATION OF INFINITE PRODUCTS

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1. Introduction

In the classical analysis there are various theorems which permit us to interchange limits and infinite sums, limits and integrals, integrals and infinite sums, etc. The infinite products as well as the infinite series play an important rôle in different branches of mathematics. It is therefore natural to study the conditions under which the infinite products and the infinite (or finite) sums are interchangeable. But there is unfortunately no theorem which permits us to do so, although many authors have investigated the infinite product identities (see [1], [2], [3], [6] and [13]) and the convergent properties of power product expansions of analytic functions (see [4], [5], [7], [8], [9], [10], [11], [12] and [14]).

For any $m, n \in \mathbf{N}$ we define

$$J_n^m = \{(j_i)_{i=1,\ldots,m} : j_i = 1, \ldots, n\}, \quad J_n = \{(j_i)_{i=1,2,\ldots} : j_i = 1, \ldots, n\},$$
$$J_\infty^m = \{(j_i)_{i=1,\ldots,m} : j_i = 1, 2, \ldots\}, J_\infty = \{(j_i)_{i=1,2,\ldots} : j_i = 1, 2, \ldots\}.$$

Empirically, according to the distributive law, we know that if a_{ij} are real numbers then

$$(a_{11} + a_{12} + a_{13})(a_{21} + a_{22} + a_{23}) = a_{11}a_{21} + a_{11}a_{22} + a_{11}a_{23}$$
$$+ a_{12}a_{21} + a_{12}a_{22} + a_{12}a_{23} + a_{13}a_{21} + a_{13}a_{22} + a_{13}a_{23},$$

i.e.,

$$\prod_{i=1}^{2}\sum_{j=1}^{3} a_{ij} = \sum_{(j_i)\in J_3^2}\prod_{i=1}^{2} a_{ij_i}.$$

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We can spontaneously generalize the above equation as follows

$$\prod_{i=1}^{n}\sum_{j=1}^{m} a_{ij} = \sum_{(j_i)\in J_m^n}\prod_{i=1}^{n} a_{ij_i} \tag{1.1}$$

when a_{ij} are real numbers. Furthermore, it is not difficult to verify

$$\prod_{i=1}^{n}\sum_{j=1}^{\infty} a_{ij} = \sum_{(j_i)\in J_\infty^n}\prod_{i=1}^{n} a_{ij_i}$$

if the series $\sum_{j=1}^{\infty} a_{ij}$ $(i = 1, ..., n)$ converge absolutely.

We may now raise some questions whether the following equations

$$\prod_{i=1}^{\infty}\sum_{j=1}^{n} a_{ij} = \sum_{(j_i)\in J_n}\prod_{i=1}^{\infty} a_{ij_i}, \tag{1.2}$$

$$\prod_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{(j_i)\in J_\infty}\prod_{i=1}^{\infty} a_{ij_i}, \tag{1.3}$$

which are generalized forms of the last equation, are also true.

DEFINITION. An infinite product $\prod_{i=1}^{\infty} a_i$ is called *convergent*, if there is a number $n \in \mathbf{N}$ such that $\lim_{N\to\infty}\prod_{i=n}^{N} a_i$ exists and is different from 0. Otherwise the infinite product is called *divergent*.

Suppose that for all $i \in \mathbf{N}$ and for $j = 1, ..., n$ $(n \geq 2)$

$$a_{ij} = n^{-1} + n^{-1}i^{-2}.$$

Then we have for all $(j_i) \in J_n$

$$\prod_{i=1}^{\infty} a_{ij_i} = 0 \ \ \text{(divergent to 0)}.$$

Hence

$$\sum_{(j_i)\in J_n}\prod_{i=1}^{\infty} a_{ij_i} = 0.$$

However, we obtain

$$\prod_{i=1}^{\infty}\sum_{j=1}^{n} a_{ij} = \prod_{i=1}^{\infty}\left(1+i^{-2}\right) \geq 1,$$

i.e., the infinite product $\prod_{i=1}^{\infty}\left(\sum_{j=1}^{n} a_{ij}\right)$ converges unconditionally and

$$\prod_{i=1}^{\infty}\sum_{j=1}^{n} a_{ij} \neq \sum_{(j_i)\in J_n}\prod_{i=1}^{\infty} a_{ij_i}.$$

Equation (1.2) is not trivial in such a sense that not every double sequence $(a_{ij})_{i=1,2,\ldots;j=1,\ldots,n}$ satisfies (1.2) (for convenience, we call $(a_{ij})_{i=1,2,\ldots;j=1,\ldots,n}$ a double sequence).

Now let $(a_{ij})_{i,j=1,2,\ldots}$ be a double sequence satisfying

$$a_{ij} = 6\pi^{-2}j^{-2}\left(1+3^{-1}i^{-2}\right).$$

Then, for all $(j_i)\in J_\infty$

$$\prod_{i=1}^{\infty} a_{ij_i} = 0 \quad \text{(divergent to 0)}.$$

Therefore

$$\sum_{(j_i)\in J_\infty}\prod_{i=1}^{\infty} a_{ij_i} = 0.$$

On the other hand,

$$\prod_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \prod_{i=1}^{\infty}\left(1+3^{-1}i^{-2}\right) \geq 1.$$

This example implies that the equation (1.3) is not trivial.

In this paper, we shall find some sufficient conditions under which (1.2) holds true and treat the equation (1.3).

2. Transformation of infinite products with finite sums as their terms

Let $p > 1$ be a fixed real number. Suppose (s_i) to be a monotone increasing sequence such that there exists a positive number α with $s_i \geq i^{1+\alpha}$ for all $i \in \mathbf{N}$. Let M_1, M_2 be positive numbers with $M_1 \geq M_2$ and let (t_i) be a bounded sequence satisfying $t_i > 1$ for all $i \in \mathbf{N}$ and $\prod_{i=1}^{\infty} t_i = M_1$. For $j = 1, ..., n$ let (a_{ij}) be a real sequence with the following properties:

$$0 < a_{i1} \leq t_i \quad \text{for all} \quad i \in \mathbf{N} \quad \text{and} \quad \prod_{i=1}^{\infty} a_{i1} = M_2, \tag{2.1}$$

$$|a_{ij}| \leq p^{-s_i} \quad (j = 2, ..., n) \quad \text{for sufficiently large} \quad i \in \mathbf{N}, \tag{2.2}$$

$$\sum_{i=1}^{\infty} \left| \sum_{j=1}^{n} |a_{ij}| - 1 \right| < \infty. \tag{2.3}$$

EXAMPLE 1. The double sequence (a_{ij}) with

$$a_{ij} = \begin{cases} 1 & \text{for } j = 1, \\ 2^{-i^2-j} & \text{for } j = 2, ..., n \end{cases}$$

satisfies the conditions of (2.1), (2.2) and (2.3).

THEOREM 1. *It holds that*

$$\prod_{i=1}^{\infty} \sum_{j=1}^{n} a_{ij} = \sum_{(j_i) \in J_n} \prod_{i=1}^{\infty} a_{ij_i}.$$

Proof. For any $N \in \mathbf{N}$ let

$$A_N = \prod_{i=1}^{N} \sum_{j=1}^{n} a_{ij}, \qquad B_N = \sum_{(j_i) \in J_n^N} \prod_{i=1}^{N} a_{ij_i},$$

$$A'_N = \prod_{i=1}^{N} \sum_{j=1}^{n} |a_{ij}|, \qquad B'_N = \sum_{(j_i) \in J_n^N} \prod_{i=1}^{N} |a_{ij_i}|.$$

Then, by (1.1), $A_N = B_N$ and $A'_N = B'_N$ for all $N \in \mathbf{N}$. Hence

(2.4) $$\lim_{N\to\infty} A_N = \lim_{N\to\infty} B_N \quad \text{and} \quad \lim_{N\to\infty} A'_N = \lim_{N\to\infty} B'_N.$$

By (2.3)

(2.5) $$\lim_{N\to\infty} A_N < \infty \quad \text{and} \quad \lim_{N\to\infty} A'_N < \infty.$$

Combining (2.4) and (2.5), it is enough to prove that

(2.6) $$\lim_{N\to\infty} B_N = \sum_{(j_i)\in J_n} \prod_{i=1}^{\infty} a_{ij_i},$$

where we assume that the summation over J_n is an ordered sum, e.g., J_n may be lexicographically ordered. For any sufficiently large $N \in \mathbf{N}$ let

(2.7)

$$\begin{aligned}\sigma_N &= \sum_{\substack{j_i=1 \text{ for} \\ \text{all } i>N}} \prod_{i=1}^{\infty} a_{ij_i} + \sum_{\substack{j_i=1 \text{ for} \\ \text{all } i\le N}} \prod_{i=1}^{\infty} a_{ij_i} + \sum_{\substack{\exists i\le N: j_i>1 \\ \exists i>N: j_i>1}} \prod_{i=1}^{\infty} a_{ij_i} \\ &= \sum_{\substack{j_i=1 \text{ for} \\ \text{all } i>N}} \prod_{i=1}^{N} a_{ij_i} \prod_{i=N+1}^{\infty} a_{i1} + \sum_{\substack{j_i=1 \text{ for} \\ \text{all } i\le N}} \prod_{i=1}^{\infty} a_{ij_i} + \sum_{\substack{\exists i\le N: j_i>1 \\ \exists i>N: j_i>1}} \prod_{i=1}^{\infty} a_{ij_i} \\ &= B_N \prod_{i=N+1}^{\infty} a_{i1} + \sigma_N^1 + \sigma_N^2\end{aligned}$$

be a reordering of $\sum_{(j_i)\in J_n} \prod_{i=1}^{\infty} a_{ij_i}$, where the first term includes $\prod_{i=1}^{\infty} a_{i1} = M_2$, and similarly, let

(2.8)

$$\begin{aligned}\sigma'_N &= \sum_{\substack{j_i=1 \text{ for} \\ \text{all } i>N}} \prod_{i=1}^{\infty} |a_{ij_i}| + \sum_{\substack{j_i=1 \text{ for} \\ \text{all } i\le N}} \prod_{i=1}^{\infty} |a_{ij_i}| + \sum_{\substack{\exists i\le N: j_i>1 \\ \exists i>N: j_i>1}} \prod_{i=1}^{\infty} |a_{ij_i}| \\ &= B'_N \prod_{i=N+1}^{\infty} |a_{i1}| + \sigma'^1_N + \sigma'^2_N\end{aligned}$$

be a reordering of $\sum_{(j_i)\in J_n}\prod_{i=1}^{\infty}|a_{ij_i}|$, where the first term includes $\prod_{i=1}^{\infty}|a_{i1}|$. With $q=\sup\{i>N : j_i>1\}$ and $r=\#\{i>N : j_i>1\}$, assume that the terms of $\sigma_N'^{1}$ are so ordered that the following estimate is possible:

$$\begin{aligned}
\sigma_N'^{1} &\leq \sum_{q=N+1}^{\infty}\sum_{r=1}^{q-N}\sum_{\substack{N<n_1<\cdots<n_r=q\\ 2\leq j_{n_t}\leq n \text{ for } t=1,\ldots,r}} M_1\prod_{i=1}^{r}|a_{n_i j_{n_i}}| \quad \text{by (2.1)}\\
&\leq M_1\sum_{q=N+1}^{\infty}\sum_{r=1}^{q-N}\sum_{\substack{N<n_1<\cdots<n_r=q\\ 2\leq j_{n_t}\leq n \text{ for } t=1,\ldots,r}}\prod_{i=1}^{r}p^{-s_{n_i}} \quad \text{by (2.2)}\\
&= M_1\sum_{q=N+1}^{\infty}\sum_{r=0}^{q-N-1}\sum_{\substack{N<n_1<\cdots<n_r<q\\ 2\leq j_{n_t}\leq n \text{ for } t=1,\ldots,r}} p^{-s_q}\prod_{i=1}^{r}p^{-s_{n_i}}\\
&\leq M_1\sum_{q=N+1}^{\infty}p^{-s_q}\sum_{r=0}^{q-N-1}\left(\binom{q-N-1}{r}(n-1)^r p^{-rs_{N+1}}\right)\\
&= M_1\sum_{q=N+1}^{\infty}p^{-s_q}\left(1+(n-1)p^{-s_{N+1}}\right)^{q-N-1}\\
&\leq M_1\sum_{q=N+1}^{\infty}p^{-s_q}\exp\left((n-1)(q-N-1)p^{-s_{N+1}}\right)
\end{aligned}$$

(2.9)

$$\leq M_1\sum_{q=N+1}^{\infty}p^{-q^{1+\alpha}}\exp\left(nqp^{-N^{1+\alpha}}\right)$$

(2.10)

$$\to 0 \quad \text{as } N\to\infty.$$

With the notation $\{i\leq N : j_i>1\}=\{n_1,\ldots,n_r\}$ we suppose that the summation of $\sigma_N'^{2}$ is such an ordered sum that the following estimate is allowed:

$$\sigma_N'^{2}\leq\sum_{r=1}^{N}\sum_{\substack{1\leq n_1<\cdots<n_r\leq N\\ 2\leq j_{n_t}\leq n \text{ for } t=1,\ldots,r}} M_1\prod_{i=1}^{r}|a_{n_i j_{n_i}}|\sum_{\substack{j_i=1 \text{ for } i=N+1\\ \text{all } i\leq N}}\prod^{\infty}|a_{ij_i}|\,.$$

By (2.2) there exists a positive number M such that $|a_{ij}| \le M$ for $j = 2, ..., n$ and for all $i \in \mathbf{N}$. Since N is sufficiently large, we have by (2.1), (2.9), (2.10) and the ratio test for convergence of series

$$\begin{aligned}
\sigma_N'^2 \le & M_1 \sum_{r=1}^{N} \left(\binom{N}{r} (n-1)^r M^r 2 M_2^{-1} \sigma_N'^1 \right) \\
\le & 2 M_1 M_2^{-1} \left((1 + (n-1)M)^N - 1 \right) \sigma_N'^1 \\
\le & 2 M_1^2 M_2^{-1} (1 + nM)^N p^{-N^{1+\alpha/2}} \\
& \sum_{q=N+1}^{\infty} p^{-(q^{1+\alpha} - N^{1+\alpha/2})} \exp\left(nqp^{-N^{1+\alpha}} \right) \\
\to & 0 \quad \text{as } N \to \infty. \qquad (2.11)
\end{aligned}$$

Combining (2.8), (2.10), (2.11), (2.4), and (2.5)

$$\lim_{N\to\infty} \sigma'_N = \lim_{N\to\infty} B'_N \prod_{i=N+1}^{\infty} |a_{i1}| = \lim_{N\to\infty} B'_N < \infty,$$

since $\prod_{i=N+1}^{\infty} |a_{i1}| \to 1$ as $N \to \infty$. Hence the series σ_N converges unconditionally if N is sufficiently large. Therefore

$$\sum_{(j_i)\in J_n} \prod_{i=1}^{\infty} a_{ij_i} = \sigma_N \qquad (2.12)$$

for sufficiently large N. On the other hand, by (2.10) and (2.11)

$$\lim_{N\to\infty} \sigma_N^1 = 0 \quad \text{and} \quad \lim_{N\to\infty} \sigma_N^2 = 0. \qquad (2.13)$$

It follows then from (2.7), (2.12), and (2.13) that

$$\sum_{(j_i)\in J_n} \prod_{i=1}^{\infty} a_{ij_i} = \lim_{N\to\infty} B_N \prod_{i=N+1}^{\infty} a_{i1} = \lim_{N\to\infty} B_N,$$

as required.

3. Interchange of infinite products and limits

Let (a_{ij}) be a double sequence satisfying $a_{ij} = 6\pi^{-2}j^{-2}$ for each $i \in \mathbf{N}$. Since $\sum_{j=1}^{\infty} j^{-2} = 6^{-1}\pi^2$, we have for any $n \in \mathbf{N}$ and for every $i \in \mathbf{N}$

$$\sum_{j=1}^{n} a_{ij} = 1 - 6\pi^{-2} \sum_{j=n+1}^{\infty} j^{-2} \quad \text{and} \quad \sum_{i=1}^{\infty} 6\pi^{-2} \sum_{j=n+1}^{\infty} j^{-2} = \infty.$$

Hence

$$\prod_{i=1}^{\infty} \sum_{j=1}^{n} a_{ij} = 0.$$

Therefore

$$\lim_{n\to\infty} \prod_{i=1}^{\infty} \sum_{j=1}^{n} a_{ij} = 0.$$

On the other hand,

$$\prod_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \prod_{i=1}^{\infty} 1 = 1.$$

In this case the infinite product symbol and the limit symbol cannot be interchanged, i.e.

$$\lim_{n\to\infty} \prod_{i=1}^{\infty} \sum_{j=1}^{n} a_{ij} \neq \prod_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}.$$

In this section we find some sufficient conditions under which the infinite product symbol and the limit symbol can be interchanged.

Let (s_n) be a positive sequence with $s_n \to \infty$ as $n \to \infty$ and let (a_{ij}) be a non-negative double sequence with the following properties:

$$\text{(3.1)} \qquad \forall i \in \mathbf{N} \;\; \exists r_i > 1 \quad 0 \leq \sum_{j=1}^{\infty} a_{ij} \leq r_i,$$

$$\text{(3.2)} \qquad \exists M > 1 \;\; : \;\; \prod_{i=1}^{\infty} r_i = M,$$

$$\text{(3.3)} \qquad \exists \alpha > 0 \;\; \exists m \in \mathbf{N} \;\; \forall i \in \mathbf{N} \; \forall n \geq m \;\; : \;\; \sum_{j=n+1}^{\infty} a_{ij} \leq s_n^{-\alpha i}.$$

EXAMPLE 2. The double sequence (a_{ij}) with

$$a_{ij} = \begin{cases} 1 + 3^{-i} & \text{for } j = 1, \\ 3^{-ij} & \text{for } j = 2, 3, \dots \end{cases}$$

satisfies (3.1), (3.2) and (3.3).

LEMMA 2 (CF. [14], THEOREM 2). *Let (a_{ij}) be the double sequence described above. Then*

$$\lim_{n\to\infty} \prod_{i=1}^{\infty} \sum_{j=1}^{n} a_{ij} = \prod_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}.$$

Proof. It follows from (3.1), (3.2) and (3.3) that for sufficiently large N and $n \geq m$

$$\begin{aligned}
&\prod_{i=1}^{N} \sum_{j=1}^{\infty} a_{ij} \\
&= \left(\sum_{j=1}^{n} a_{1j} + \sum_{j=n+1}^{\infty} a_{1j}\right)\left(\sum_{j=1}^{n} a_{2j} + \sum_{j=n+1}^{\infty} a_{2j}\right) \cdot \ldots \cdot \left(\sum_{j=1}^{n} a_{Nj} + \sum_{j=n+1}^{\infty} a_{Nj}\right) \\
&= \prod_{i=1}^{N} \sum_{j=1}^{n} a_{ij} + \sum_{i_1=1}^{N} \left(\sum_{j=n+1}^{\infty} a_{i_1 j}\right) \prod_{\substack{i=1 \\ i \neq i_1}}^{N} \sum_{j=1}^{n} a_{ij} \\
&\quad + \sum_{i_1=1}^{N} \sum_{i_2=i_1+1}^{N} \left(\sum_{j=n+1}^{\infty} a_{i_1 j}\right)\left(\sum_{j=n+1}^{\infty} a_{i_2 j}\right) \prod_{\substack{i=1 \\ i \neq i_1, i_2}}^{N} \sum_{j=1}^{n} a_{ij} + \cdots + \\
&\quad + \sum_{i_1=1}^{N} \sum_{i_2=i_1+1}^{N} \cdots \sum_{i_N=i_{N-1}+1}^{N} \left(\sum_{j=n+1}^{\infty} a_{i_1 j}\right)\left(\sum_{j=n+1}^{\infty} a_{i_2 j}\right) \cdot \ldots \cdot \left(\sum_{j=n+1}^{\infty} a_{i_N j}\right) \\
&\leq \prod_{i=1}^{N} \sum_{j=1}^{n} a_{ij} + M \sum_{i_1=1}^{N} \sum_{j=n+1}^{\infty} a_{i_1 j} + M \sum_{i_1=1}^{N} \sum_{i_2=i_1+1}^{N} \left(\sum_{j=n+1}^{\infty} a_{i_1 j}\right)\left(\sum_{j=n+1}^{\infty} a_{i_2 j}\right) \\
&\quad + \cdots + \\
&\quad + M \sum_{i_1=1}^{N} \sum_{i_2=i_1+1}^{N} \cdots \sum_{i_N=i_{N-1}+1}^{N} \left(\sum_{j=n+1}^{\infty} a_{i_1 j}\right)\left(\sum_{j=n+1}^{\infty} a_{i_2 j}\right) \cdot \ldots \cdot \left(\sum_{j=n+1}^{\infty} a_{i_N j}\right) \\
&\leq \prod_{i=1}^{N} \sum_{j=1}^{n} a_{ij} + M \sum_{i=1}^{\infty} (s_n^{\alpha} - 1)^{-n(n+1)/2}.
\end{aligned}$$

By letting $N \to \infty$ it follows that

$$\prod_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \le \prod_{i=1}^{\infty}\sum_{j=1}^{n} a_{ij} + O(s_n^{-\alpha})$$

and then by letting $n \to \infty$ again

$$\prod_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \le \lim_{n\to\infty} \prod_{i=1}^{\infty}\sum_{j=1}^{n} a_{ij}.$$

The opposite inequality is obvious, since (a_{ij}) is a non-negative double sequence.

4. Transformation of infinite products with infinite sums as their terms

Let $p > 1$ be fixed. Suppose (s_i) to be a monotone increasing sequence such that there exists a positive number α with $s_i \ge i^{1+\alpha}$ for all $i \in \mathbf{N}$. Let M_1, M_2 be positive numbers with $M_1 \ge M_2$ and let (t_i) be a bounded sequence satisfying $t_i > 1$ for all $i \in \mathbf{N}$ and $\prod_{i=1}^{\infty} t_i = M_1$. Let (a_{ij}) be a non-negative double sequence with the following properties:

$$0 < a_{i1} \le t_i \quad \text{for all } i \in \mathbf{N} \text{ and } \prod_{i=1}^{\infty} a_{i1} = M_2, \tag{4.1}$$

$$\exists \beta > 0 \;\; \forall i \ge 1 \;\; \forall j > 1 \; : \; 0 \le a_{ij} \le p^{-\beta s_i j}, \tag{4.2}$$

$$\sum_{i=1}^{\infty} \Big| \sum_{j=1}^{\infty} a_{ij} - 1 \Big| < \infty. \tag{4.3}$$

EXAMPLE 3. The double sequence (a_{ij}) with

$$a_{ij} = \begin{cases} 1 + 2^{-i^2} & \text{for } j = 1, \\ 2^{-i^2 j} & \text{for } j = 2, 3, \ldots \end{cases}$$

satisfies (4.1), (4.2) and (4.3).

THEOREM 3.

$$\prod_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \lim_{n\to\infty} \sum_{(j_i)\in J_n} \prod_{i=1}^{\infty} a_{ij_i}.$$

Proof. For all sufficiently large $n \in \mathbf{N}$ let

$$A_n = \prod_{i=1}^{\infty}\sum_{j=1}^{n} a_{ij} \quad \text{and} \quad B_n = \sum_{(j_i)\in J_n} \prod_{i=1}^{\infty} a_{ij_i}.$$

It follows from (4.2) and (4.3) that for all sufficiently large $n \in \mathbf{N}$

$$\sum_{i=1}^{\infty} \Big|\sum_{j=1}^{n} a_{ij} - 1\Big| < \infty.$$

Therefore, it follows from Theorem 1 that $A_n = B_n$ for all sufficiently large $n \in \mathbf{N}$. So

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n. \tag{4.4}$$

On account of Lemma 2, (4.4) and (4.3)

$$\prod_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{(j_i)\in J_n} \prod_{i=1}^{\infty} a_{ij_i} < \infty.$$

Open problem. Under the conditions described above, does it hold true

$$\lim_{n\to\infty} \sum_{(j_i)\in J_n} \prod_{i=1}^{\infty} a_{ij_i} = \sum_{(j_i)\in J_\infty} \prod_{i=1}^{\infty} a_{ij_i} \ ?$$

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