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ASYMPTOTIC BEHAVIOR OF LEBESGUE MEASURES OF CANTOR SETS ARISING IN THE DYNAMICS OF TANGENT FAMILY $T_\alpha(\theta) = \alpha\tan(\theta/2)$

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1. Introduction

Let $0 < \alpha < 2$ and let $T_\alpha(\theta) = \alpha\tan(\theta/2)$. T_α has an attractive fixed point at $\theta = 0$. We denote by $C(\alpha)$ the set of points in $I = [-\pi, \pi]$ which are not attracted to $\theta = 0$ by the succesive iterations of T_α. That is, $C(\alpha)$ is the set of points in I where the dynamics of T_α is chaotic. It is also related to the Julia set of the family of point-mass singular inner functions $\exp(\alpha\frac{1-z}{1+z})$ and is shown to be a Cantor set, a closed, totally disconnected, perfect subset of I in [KK1,KK2]. Since T_α has a slope $\alpha/2$ at $\theta = 0$ which is less than one, it has two more fixed points in I other than the attractive fixed point at $\theta = 0$. They are symmetrically located at $-\theta_0(\alpha)$ and $\theta_0(\alpha)$. See Figure 1. We denote the interval $(-\theta_0(\alpha), \theta(\alpha))$ by $E_0(\alpha)$. From the graphical analysis, the points on the interval $E_0(\alpha)$ are shown to be attracted to the attractive fixed point at $\theta = 0$ under T_α. Therefore, $C(\alpha) \subset I\backslash E_0(\alpha)$. Since the Lebesgue measure $|E_0(\alpha)|$ of $E_0(\alpha)$ tends to 2π as $\alpha \to 0$, the Lebesgue measure $|C(\alpha)|$ of $C(\alpha)$ tends to 0 as $\alpha \to 0$. We can say more precisely on the asymptotic behavior of $|C(\alpha)|$ as $\alpha \to 0$ in the following theorem.

THEOREM.
(a) $|C(\alpha)| = O(\alpha^5)$ *as* $\alpha \to 0$, *analytically.*
(b) $|C(\alpha)| = O(\alpha^7)$ *as* $\alpha \to 0$, *by use of MATHEMATICA.*

It is a reasonable conjecture that $|C(\alpha)|$ is of infinite order as $\alpha \to 0$.

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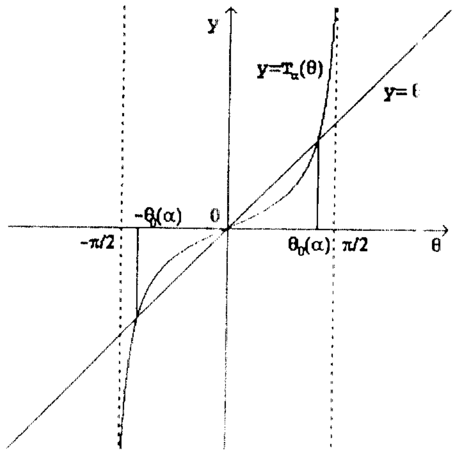

Figure 1

2. Analytical proof

Let $E_1(\alpha)$ be the set of points θ in $I\backslash E_0(\alpha)$ such that $T_\alpha(\theta) \in E_0(\alpha)$ (mod 2π). Then $E_1(\alpha) = \bigcup\{E_1^{(k_1)} : k_1 = \pm1, \pm2, \cdots\}$, where

$$E_1^{(k_1)} = \{\theta \in I : T_\alpha(\theta) \in E_0(\alpha) + 2k_1\pi\}.$$

Let $E_2(\alpha)$ be the set of points θ in $I\backslash(E_0(\alpha) \cup E_1(\alpha))$ such that $T_\alpha^2(\theta) \in E_0(\alpha)$ (mod 2π). Then $E_2(\alpha)$ is given as

$$E_2(\alpha) = \bigcup\{E_2^{(k_1,k_2)} : k_1 = \pm1, \pm2, \cdots; k_2 = 0, \pm1, \pm2, \cdots\},$$

where

$$E_2^{(k_1,k_2)} = \{\theta \in I : T_\alpha(\theta) \in E_1^{(k_1)}(\alpha) + 2k_2\pi\}.$$

Inductively, we let $E_n(\alpha)$ be the set of points θ in $I\backslash \bigcup_{j=1}^{n-1} E_j(\alpha)$ such that $T_\alpha^n(\theta) \in E_0(\alpha)$ (mod 2π). Then

$$E_n(\alpha) = \bigcup\{E_n^{(k_1,k_2,\cdots,k_n)} : k_1 = \pm1, \pm2, \cdots;$$
$$k_2, \cdots, k_n = 0, \pm1, \pm2, \cdots\},$$

where

$$E_n^{(k_1,k_2,\cdots,k_n)} = \{\theta \in I : T_\alpha(\theta) \in E_{n-1}^{(k_1,\cdots,k_{n-1})} + 2k_n\pi\}.$$

Then clearly $C(\alpha)$ is given by

$$C(\alpha) = I \setminus \bigcup_{n=0}^{\infty} E_n(\alpha).$$

For an n-tuple $(k_1, k_2, \cdots, k_n)$ of nonzero integer k_1 and intege rs $k_2, \cdots, k_n$, we set

$$\theta_1^{(0)} = -\theta_0(\alpha)\ ,\ \theta_2^{(0)} = \theta_0(\alpha),$$

$$\theta_1^{(1)} = 2\tan\left(\frac{2k_1\pi + \theta_1^{(0)}}{\alpha}\right)\ ,\ \theta_2^{(1)} = 2\tan\left(\frac{2k_1\pi + \theta_2^{(0)}}{\alpha}\right),$$

and, inductively,

$$\theta_1^{(n)} = 2\tan^{-1}\left(\frac{2k_n\pi + \theta_1^{(n-1)}}{\alpha}\right)\ ,\ \theta_2^{(n)} = 2\tan^{-1}\left(\frac{2k_n\pi + \theta_2^{(n-1)}}{\alpha}\right),$$

for $n = 1, 2, \ldots$. Then $E_n^{(k_1,\cdots,k_n)} = (\theta_1^{(n)}, \theta_2^{(n)})$. Of course, $\theta_i^{(n)}$'s depend on the choice of $k_1, k_2, \ldots, k_n$ but we suppressed its dependence for a notational simplicity.

Now, we estimate $|E_n(\alpha)|$ up to order 4 in its Taylor expansion at $\alpha = 0$. The following identities are easy to obtain.

2.1 Lemma.

(i) $\displaystyle\sum_{k\neq 0} \frac{1}{(2k)^2 - 1} = 1$

(ii) $\displaystyle\sum_{k\neq 0} \left(\frac{1}{(2k)^2 - 1}\right)^2 = \frac{\pi^2}{8} - 1$

(iii) $\displaystyle\sum_{k\neq 0} \left(\frac{1}{(2k)^2 - 1}\right)^3 = 1 - \frac{3\pi^2}{32}$

(iv) $\displaystyle\sum_{k\neq 0} \left(\frac{1}{(2k)^2 - 1}\right)^4 = \frac{\pi^4}{384} + \frac{5\pi^2}{64} - 1$

(v) $\displaystyle\sum_{k\neq 0} \frac{k^2}{((2k)^2 - 1)^3} = \frac{\pi^2}{128}$

(vi) $\displaystyle\sum_{-\infty}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{4}$

(vii) $\displaystyle\sum_{-\infty}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{48}$

To begin with, the leading terms of $|E_n(\alpha)|$ is generally given by

2.2 LEMMA.

$$|E_n(\alpha)| = \frac{8}{\pi}\left(\frac{\alpha}{2}\right)^n + O(\alpha^{n+1}).$$

Proof. For an integer $k_1 \neq 0$

$$\begin{aligned}
|E_1^{(k_1)}| &= 2\tan^{-1}\left(\frac{2k_1\pi + \theta_2^{(0)}}{\alpha}\right) - 2\tan^{-1}\left(\frac{2k_1\pi + \theta_1^{(0)}}{\alpha}\right) \\
&= 2\tan^{-1}\left(\frac{2\alpha\theta_0(\alpha)}{(2k_1\pi)^2 - \theta_0^2(\alpha) + \alpha^2}\right) \\
&= 2\pi \cdot \frac{2\alpha}{\pi^2((2k_1)^2 - 1)} + O(\alpha^2).
\end{aligned}$$

For an integer k_2,

$$\begin{aligned}
|E_2^{(k_1,k_2)}| &= 2\tan^{-1}\left(\frac{2k_2\pi + \theta_2^{(1)}}{\alpha}\right) - 2\tan^{-1}\left(\frac{2k_2\pi + \theta_1^{(1)}}{\alpha}\right) \\
&= 2\tan^{-1}\left(\frac{\alpha(\theta_2^{(1)} - \theta_1^{(1)})}{(2k_2\pi + \theta_1^{(1)})(2k_2\pi + \theta_2^{(1)}) + \alpha^2}\right) \\
&= 2\pi \frac{2\alpha}{\pi^2((2k_1)^2 - 1)} \frac{2\alpha}{(2k_2 + 1)^2\pi^2} + O(\alpha^3).
\end{aligned}$$

Thus we have inductively

$$\begin{aligned}
&|E_n^{(k_1,\cdots,k_n)}| \\
&= 2\pi \frac{2\alpha}{\pi^2((2k_1)^2 - 1)} \frac{2\alpha}{\pi^2(2k_2 + 1)^2} \cdots \frac{2\alpha}{\pi^2(2k_n + 1)^2} + O(\alpha^{n+1}).
\end{aligned}$$

Therefore, by Lemma 2.1, we get

$$\begin{aligned}
|E_n(\alpha)| &= \sum |E_n^{(k_1,k_2,\dots,k_n)}| \\
&= \frac{2^{n+1}\alpha^n}{\pi^{2n-1}}\left(\sum_{k_1\neq 0} \frac{1}{(2k_1)^2 - 1}\right)\left(\sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^2}\right)^{n-1} + O(\alpha^{n+1}) \\
&= \frac{8}{\pi}\left(\frac{\alpha}{2}\right)^n + O(\alpha^{n+1}),
\end{aligned}$$

where the first sum runs over all nonzero integers k_1 and all intergers $k_2, k_3, \cdots, k_n$.

The Taylor expansion of $|E_n(\alpha)|$ near $\alpha = 0$ up to order 4 is given as follows.

2.3. LEMMA.

(i) $|E_0(\alpha)| = 2\pi-\frac{4}{\pi}\alpha-\frac{8}{\pi^3}\alpha^2+(\frac{4}{3\pi^3}-\frac{32}{\pi^5})\alpha^3+(\frac{32}{3\pi^5}-\frac{160}{\pi^7})\alpha^4+O(\alpha^5)$,

(ii) $|E_1(\alpha)| = \frac{4}{\pi}\alpha - (\frac{2}{\pi} - \frac{8}{\pi^3})\alpha^2 - (\frac{16}{3\pi^3} - \frac{32}{\pi^5})\alpha^3 + (\frac{1}{6\pi} - \frac{80}{3\pi^5} + \frac{160}{\pi^7})\alpha^4 + O(\alpha^5)$,

(iii) $|E_2(\alpha)| = \frac{2}{\pi}\alpha^2 - (\frac{1}{\pi} - \frac{4}{\pi^3})\alpha^3 - (\frac{1}{6\pi} + \frac{2}{\pi^3} - \frac{16}{\pi^5})\alpha^4 + O(\alpha^5)$,

(iv) $|E_3(\alpha)| = \frac{1}{\pi}\alpha^3 - (\frac{1}{2\pi} - \frac{2}{\pi^3})\alpha^4 + O(\alpha^5)$,

(v) $|E_4(\alpha)| = \frac{1}{2\pi}\alpha^4 + O(\alpha^5)$.

Proof. From the relation $\theta_0(\alpha) = \alpha\tan(\theta_0(\alpha)/2)$ and $\lim_{\alpha\to 0}\theta_0(\alpha) = \pi$, we have the following Taylor expansion of $\theta_0(\alpha)$ near $\alpha = 0$.

$$\theta_0(\alpha) = \pi - \frac{2}{\pi}\alpha - \frac{4}{\pi^3}\alpha^2 + \left(\frac{2}{3\pi^3} - \frac{16}{\pi^5}\right)\alpha^3 + \left(\frac{16}{3\pi^5} - \frac{80}{\pi^7}\right)\alpha^4 + O(\alpha^5).$$

Therefore,

$$|E_0(\alpha)| = 2\pi-\frac{4}{\pi}\alpha-\frac{8}{\pi^3}\alpha^2+\left(\frac{4}{3\pi^3} - \frac{32}{\pi^5}\right)\alpha^3+\left(\frac{32}{3\pi^5} - \frac{160}{\pi^7}\right)\alpha^4+O(\alpha^5).$$

For a nonzero integer k_1, we have the following expansion by a routine calculation:

$$\begin{aligned}
|E_1^{(k_1)}| &= \theta_2^{(1)} - \theta_1^{(1)} = 2\tan^{-1}\left(\frac{2k_1\pi + \theta_2^{(0)}}{\alpha}\right) - 2\tan^{-1}\left(\frac{2k_1\pi + \theta_1^{(0)}}{\alpha}\right) \\
&=2\tan^{-1}\left(\frac{2\alpha\theta_0(\alpha)}{(2k_1\pi)^2 - \theta_0^2(\alpha) + \alpha^2}\right) \\
&=2\frac{2\alpha\theta_0(\alpha)}{(2k_1\pi)^2 - \theta_0^2(\alpha) + \alpha^2} - \frac{2}{3}\left(\frac{2\alpha\theta_0(\alpha)}{(2k_1\pi)^2 - \theta_0^2(\alpha) + \alpha^2}\right)^3 + O(\alpha^5) \\
&=\frac{4\alpha}{\pi}\frac{1}{(2k_1)^2 - 1} - \left\{\frac{8}{\pi^3((2k_1)^2 - 1)} + \frac{16}{\pi^3}\left(\frac{1}{(2k_1)^2 - 1}\right)^2\right\}\alpha^2 \\
&\quad - \left\{\left(\frac{16}{3\pi^3} - \frac{64}{\pi^5}\right)\left(\frac{1}{(2k_1)^2 - 1}\right)^3 + \left(\frac{4}{\pi^3} - \frac{16}{\pi^5}\right)\left(\frac{1}{(2k_1)^2 - 1}\right)^2\right. \\
&\quad \left. + \frac{16}{\pi^5}\frac{1}{(2k_1)^2 - 1}\right\}\alpha^3 - \left\{\left(\frac{256}{\pi^7} - \frac{64}{\pi^5}\right)\left(\frac{1}{(2k_1)^2 - 1}\right)^4 - \frac{64}{\pi^5}\left(\frac{1}{(2k_1)^2 - 1}\right)^3\right. \\
&\quad \left. - \left(\frac{40}{3\pi^5} + \frac{32}{\pi^7}\right)\left(\frac{1}{(2k_1)^2 - 1}\right)^2 - \left(\frac{8}{3\pi^5} - \frac{64}{\pi^7}\right)\frac{1}{(2k_1)^2 - 1}\right\}\alpha^4 + O(\alpha^5).
\end{aligned}$$

Therefore, we get by use of Lemma 2.1

$$|E_1(\alpha)| = \sum |E_1^{(k_1)}| = \frac{4}{\pi}\alpha - \left(\frac{2}{\pi} - \frac{8}{\pi^3}\right)\alpha^2 - \left(\frac{16}{3\pi^2} + \frac{32}{\pi^5}\right)\alpha^3 + \left(\frac{1}{6\pi} - \frac{80}{3\pi^5} + \frac{160}{\pi^7}\right)\alpha^4 + O(\alpha^5),$$

where the sum runs over all nonzero integers k_1.

Next, for an integer k_2, we have

$$\begin{aligned}
&|E_2^{(k_1,k_2)}| \\
&= \theta_2^{(2)} - \theta_1^{(2)} = 2\tan^{-1}\left(\frac{2k_2\pi + \theta_2^{(1)}}{\alpha}\right) - 2\tan^{-1}\left(\frac{2k_2\pi + \theta_1^{(1)}}{\alpha}\right) \\
&= 2\tan^{-1}\left(\frac{\alpha(\theta_2^{(1)} - \theta_1^{(1)})}{(2k_2\pi + \theta_1^{(1)})(2k_2\pi + \theta_2^{(1)}) + \alpha^2}\right) \\
&= 2\frac{\alpha(\theta_2^{(1)} - \theta_1^{(1)})}{(2k_2\pi + \theta_1^{(1)})(2k_2\pi + \theta_2^{(1)}) + \alpha^2} + O(\alpha^5) \\
&= \frac{8\alpha^2}{\pi^3((2k_1)^2 - 1)(2k_2+1)^2} + \alpha^3\left\{\frac{64k_1}{\pi^5((2k_1)^2 - 1)^2(2k_2+1)^3}\right. \\
&\quad \left. - \frac{16}{\pi^5((2k_1) - 1)(2k_2+1)^2} - \frac{32}{\pi^5((2k_1)^2 - 1)^2(2k_2+1)^2}\right\} \\
&\quad + \alpha^4\left\{-\frac{128k_1}{\pi^7((2k_1)^2 - 1)^2(2k_2+1)^3} - \frac{256k_1}{\pi^7((2k_1)^2 - 1)^3(2k_2+1)^3}\right. \\
&\quad + \left(\frac{16}{\pi^4} - \frac{4}{3\pi^2}\right)\frac{8}{\pi^3((2k_1)^2 - 1)^3(2k_2+1)^2} + \frac{8}{\pi^3((2k_1)^2 - 1)^2(2k_2+1)^2} \\
&\quad \left(\frac{4}{\pi^4} - \frac{1}{\pi^2}\right) - \frac{32}{\pi^7}\frac{1}{((2k_1)^2 - 1)(2k_2+1)^2} + \frac{512k_1^2}{\pi^7((2k_1)^2 - 1)^3(2k_2+1)^4} \\
&\quad - \frac{8}{\pi^5((2k_1)^2 - 1)(2k_2+1)^4} - \frac{32}{\pi^7((2k_1)^2 - 1)^2(2k_2+1)^4} \\
&\quad \left. + \frac{32}{\pi^7((2k_1)^2 - 1)(2k_1+1)^2(2k_2+1)^3} - \frac{32}{\pi^7((2k_1)^2 - 1)(2k_1-1)^2(2k_2+1)^3}\right\} \\
&\quad + O(\alpha^5).
\end{aligned}$$

Therefore, we have by use of Lemma 2.1

$$\begin{aligned}
|E_2(\alpha)| &= \sum |E_1^{(k_1,k_2)}| \\
&= \frac{2}{\pi}\alpha^2 - \left(\frac{1}{\pi} - \frac{4}{\pi^3}\right)\alpha^3 - \left(\frac{1}{6\pi} + \frac{2}{\pi^3} - \frac{16}{\pi^5}\right)\alpha^4 + O(\alpha^5),
\end{aligned}$$

where the sum runs over all nonzero integers k_1 and all integers k_2.

Next, for integers k_2 and k_3, we compute

$$
\begin{aligned}
|E_3^{(k_1,k_2,k_3)}| &= \theta_2^{(3)} - \theta_1^{(3)} = 2\tan^{-1}\left(\frac{\alpha(\theta_2^{(2)} - \theta_1^{(2)})}{(2k_2\pi + \theta_1^{(2)})(2k_2\pi + \theta_2^{(2)}) + \alpha^2}\right) \\
&= 2\frac{\alpha(\theta_2^{(2)} - \theta_1^{(2)})}{(2k_2\pi + \theta_1^{(2)})(2k_2\pi + \theta_2^{(1)}) + \alpha^2} + O(\alpha^5) \\
&= \frac{16}{\pi^5}\frac{1}{(2k_1)^2 - 1}\left(\frac{1}{2k_2+1}\right)^2\left(\frac{1}{2k_3+1}\right)^2\alpha^3 \\
&\quad + \frac{32}{\pi^7}\left\{\frac{4k_1}{((2k_1)^2-1)^2}\left(\frac{1}{2k_2+1}\right)^3\left(\frac{1}{2k_3+1}\right)^2\right. \\
&\quad - \frac{1}{(2k_1)^2 - 1}\left(\frac{1}{2k_2+1}\right)^2\left(\frac{1}{2k_3+1}\right)^2 \\
&\quad - \frac{2}{((2k_1)^2 - 1)^2}\left(\frac{1}{2k_2+1}\right)^2\left(\frac{1}{2k_3+1}\right)^2 \\
&\quad \left. + \frac{2}{(2k_1)^2 - 1}\left(\frac{1}{2k_2+1}\right)^3\left(\frac{1}{2k_3+1}\right)^3\right\}\alpha^4 + O(\alpha^5).
\end{aligned}
$$

Therefore, we have by the use of Lemma 2.1

$$
\begin{aligned}
|E_3(\alpha)| &= \sum |E_3^{(k_1,k_2,k_3)}| \\
&= \frac{1}{\pi}\alpha^3 - \left(\frac{1}{2\pi} - \frac{2}{\pi^3}\right)\alpha^4 + O(\alpha^5),
\end{aligned}
$$

where the sum runs over all nonzero integers k_1 and all integers k_2, k_3. Finally by Lemma 2.2, we have

$$|E_4(\alpha)| = \frac{1}{2\pi}\alpha^4 + O(\alpha^5).$$

2.4 Proof of Theorem (a). By Lemma 2.3, we have

$$|\bigcup_{n=0}^{4} E_n(\alpha)| = \sum_{n=0}^{4} |E_n(\alpha)| = 2\pi + O(\alpha^5)$$

Since $|E_n(\alpha)| = \frac{8}{\pi}\left(\frac{\alpha}{2}\right)^2 + O(\alpha^{n+1})$ by Lemma 2.2, we have

$$|\bigcup_{n=5}^{\infty} E_n(\alpha)| = \sum_{n=5}^{\infty} |E_n(\alpha)| = O(\alpha^5).$$

Therefore, we have

$$|\bigcup_{n=0}^{\infty} E_n(\alpha)| = \sum_{n=0}^{4} |E_n(\alpha)| + \sum_{n=5}^{\infty} |E_n(\alpha)| = 2\pi + O(\alpha^5),$$

and hence we have our main result,

$$|C(\alpha)| = O(\alpha^5), \quad \text{as} \quad \alpha \to 0.$$

3. Proof by mathematica

We used the MATHEMATICA to obtain the Taylor expansion of $|E_n(\alpha)|$ up to order 6. They are given as follows :

(i) $$|E_0(\alpha)| = 2\pi - \frac{4}{\pi}\alpha - \frac{8}{\pi^3}\alpha^2 + (\frac{4}{3\pi^3} - \frac{32}{\pi^5})\alpha^3 + (\frac{32}{3\pi^5} - \frac{160}{\pi^7})\alpha^4 - (\frac{4}{5\pi^5} - \frac{80}{\pi^7} + \frac{896}{\pi^9})\alpha^5 - (\frac{184}{15\pi^7} - \frac{1792}{\pi^9} + \frac{5376}{\pi^{11}})\alpha^6 + O(\alpha^7),$$

(ii) $$|E_1(\alpha)| = \frac{4}{\pi}\alpha - (\frac{2}{\pi} - \frac{8}{\pi^3})\alpha^2 - (\frac{16}{3\pi^3} - \frac{32}{\pi^5})\alpha^3 + (\frac{1}{6\pi} - \frac{80}{3\pi^5} + \frac{160}{\pi^7})\alpha^4 + (\frac{1}{3\pi^3} + \frac{32}{15\pi^5} - \frac{160}{\pi^7} + \frac{896}{\pi^9})\alpha^5 - (\frac{1}{60\pi} - \frac{1}{6\pi^3} - \frac{4}{3\pi^5} - \frac{424}{15\pi^7} + \frac{3136}{3\pi^9} - \frac{5376}{\pi^{11}})\alpha^6 + O(\alpha^7),$$

(iii) $$|E_2(\alpha)| = \frac{2}{\pi}\alpha^2 - (\frac{1}{\pi} - \frac{4}{\pi^3})\alpha^3 - (\frac{1}{6\pi} + \frac{2}{\pi^3} - \frac{16}{\pi^5})\alpha^4 + (\frac{1}{12\pi} - \frac{1}{3\pi^3} - \frac{28}{3\pi^5} + \frac{80}{\pi^7})\alpha^5 + (\frac{1}{60\pi} - \frac{2}{3\pi^5} - \frac{56}{\pi^7} + \frac{448}{\pi^9})\alpha^6 + O(\alpha^7),$$

(iv) $$|E_3(\alpha)| = \frac{1}{\pi}\alpha^3 - (\frac{1}{2\pi} - \frac{2}{\pi^3})\alpha^4 - (\frac{1}{12\pi} + \frac{1}{\pi^3} - \frac{8}{\pi^5})\alpha^5 + (\frac{1}{24\pi} - \frac{1}{6\pi^3} - \frac{14}{3\pi^5} + \frac{4}{\pi^7})\alpha^6 + O(\alpha^7),$$

(v) $$|E_4(\alpha)| = \frac{1}{2\pi}\alpha^4 - (\frac{1}{4\pi} - \frac{1}{\pi^3})\alpha^5 - (\frac{1}{24\pi} + \frac{1}{2\pi^3} - \frac{4}{\pi^5})\alpha^6 + O(\alpha^7),$$

(vi) $$|E_5(\alpha)| = \frac{1}{4\pi}\alpha^5 - (\frac{1}{8\pi} - \frac{1}{2\pi^3})\alpha^6 + O(\alpha^7),$$

(vii) $$|E_6(\alpha)| = \frac{\alpha^6}{8\pi} + O(\alpha^7),$$

Since $|E_n(\alpha)| = \frac{8}{\pi}(\frac{\alpha}{2})^n + O(\alpha^{n+1})$ by Lemma 2.2, we have

$$|\bigcup_{n=7}^{\infty} E_n(\alpha)| = \sum_{n=7}^{\infty} |E_n(\alpha)| = O(\alpha^7).$$

Therefore, we have

$$|\bigcup_{n=0}^{\infty} E_n(\alpha)| = \sum_{n=0}^{6} |E_n(\alpha)| + \sum_{n=7}^{\infty} |E_n(\alpha)| = 2\pi + O(\alpha^7).$$

and hence,

$$|C(\alpha)| = O(\alpha^7), \text{ as } \alpha \to 0.$$

4. Concluding remarks

The Cantor set $C(\alpha)$ is atypical as compared with the classical Cantor set. When we encountered with the Cantor set $C(\alpha)$, we wondered whether the Lebesgue measure of $C(\alpha)$ is positive for all $0 < \alpha < 2$ or not. We could only prove that $|C(\alpha)| = O(\alpha^5)$ and conjecture that it is of infinite order as $\alpha \to 0$. Also it would be an interesting question to know the Hausdorff dimension of the set $C(\alpha)$.

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