Pusan Kyŏngnam Math. J. 12(1996), No. 1, pp. 125–145

ABSTRACT FUNCTIONAL EVOLUTIONS IN GENERAL BANACH SPACES

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1. Introduction and preliminaries

Let X be a real Banach space with norm $\|\cdot\|$. We let C denote the space of all continuous functions $f : [-r, 0] \to X$ for a fixed $r > 0$. For $f \in C$, $\|f\|_C = \sup_{-r\leq s\leq 0} \|f(s)\|$.

We consider the abstract functional evolutions of the type

$$(\text{FDE} : \phi) \qquad \begin{cases} x'(t) + A(t, x_t)x(t) \ni G(t, x_t), & t \in [0, T], \\ x_0 = \phi, \quad -r \leq t \leq 0 \end{cases}$$

in a general Banach space, where for a function $f : [-r, T] \to X$, $f_t(s) = f(t + s)$, $t \in [0, T]$, $s \in [-r, 0]$ with a positive constant T.

An operator $A : D \subset X \to 2^X$ is called "accretive" if

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\|$$

for every $\lambda > 0$ and every $[x_1, y_1], [x_2, y_2] \in A$. It is called "$m$-accretive" if it is accretive and $R(I + \lambda A) = X$ for all $\lambda > 0$. If A is m-accretive, we set

$$|Ax| = \lim_{\lambda\downarrow 0} \|A_\lambda x\|, \quad x \in X,$$

where $A_\lambda = (I - J_\lambda)/\lambda$ with $J_\lambda = (I + \lambda A)^{-1}$. We also set

$$\hat{D} = \{x \in X : |Ax| < \infty\}.$$

Received May 30, 1996

†The present studies were supported in part by the Basic Science Research Institute Program, Ministry of Education, Korea, 1994, Project No BSRI-94-1410.

It is known that $D(A) \subset \hat{D}(A) \subset \overline{D(A)}$. For other properities of these operators, the reader is referred to Barbu [1], Crandall [2], Crandall and Pazy [3] and Evans [4].

Tanaka [12] has recently obtained the existence of a unique limit solution of the abstract nonlinear functional evolution problem of the type

$$x'(t) + A(t)x(t) \ni G(t, x_t), \quad t \in [0, T], \quad x_0 = \phi$$

in a general Banach space by constructing the "lines" which satisfy certain approximate discrete scheme. The solution is obtained from the uniform limit of the "lines". Kartsatos and Parrott [10] also have the similar results with different method. For the operator $A(t, x_t)$, Kartsatos and Parrott [8], Kartsatos [7] have studied by use of fixed point theory and Crandall and Pazy's result [3].

The following conditions will be used in the sequel.

(A.1) For each $(t, \psi) \in [0, T] \times C$, $A(t, \psi) : D(A(t, \psi)) \subset X \to 2^X$ is m-accretive in X, where $D(A(t, \psi))$ is only dependent on t. We denote $D(A(t, \psi)) = D(t)$.

(A.2) For each $t, s \in [0, T]$, $\psi_1, \psi_2 \in C$, and $v \in X$,

$$\begin{aligned}&\|A_\lambda(t, \psi_1)v - A_\lambda(s, \psi_2)v\| \\ &\quad \le L_0(\|v\|)[|t - s|(1 + \|A_\lambda(s, \psi_2)v\|) + \|\psi_1 - \psi_2\|_C]\end{aligned}$$

where $L_0 : \mathcal{R}^+ \to \mathcal{R}^+ = [0, \infty)$ is nondecreasing, continuous function.

(A.3) For $t, s \in [0, T]$, and $\psi, \psi_1, \psi_2 \in C$,

$$\|G(t, \psi_1) - G(t, \psi_2)\| \le k_1\|\psi_1 - \psi_2\|_C,$$
$$\|G(t, \psi) - G(s, \psi)\| \le L_1(\|\psi\|_C)|t - s|,$$

where k_1 is a positive constant and $L_1 : \mathcal{R}^+ \to \mathcal{R}^+$ is nondecreasing, continuous function.

(A.4) ϕ is a given Lipschitz continuous function with Lipschitz constant k_0 on $[-r, 0]$.

By virtue of (A.2), it is known that $\hat{D}(A(t, \psi))$ is independent of $(t, \psi) \in [0, T] \times C$. (See Evans [4].) We denote by $\hat{D} \equiv \hat{D}(A(t, \psi))$.

The main purpose of this paper is to obtain a "generalized solution " of (FDE:ϕ) with direct method. When the functional term in A and G is fixed, (FDE:ϕ) is converted a very well known evolution problem. Then we employ the Banach contraction principle to get a local generalized solution.

We define a set E by

$$E = \{u : [-r, T] \to X \mid u(t) \text{ is continuous, } u(t) = \phi(t) \text{ for } t \in [-r, 0] \\ \text{and } \|u(t_1) - u(t_2)\| \leq M|t_1 - t_2| \text{ for } t_1, t_2 \in [0, T]\},$$

where

$$M > \max\{k_0, (|A(0,\phi)\phi(0)| + \|G(0,\phi)\|)e\}$$

is a constant. Clearly, $E \neq \phi$ since the function $u(t)$ defined by $u(t) = \phi(t)$ for $t \in [-r, 0]$, and $u(t) = \phi(0)$ for $t \in [0, T]$ belongs to E. Moreover, the set E is a complete metric space with supremum norm $\|\cdot\|_{[-r,T]}$.

2. Main results

In the following discussion, we assume that the hypotheses (A.1)–(A.4) hold and $\phi(0) \in \hat{D}$. Let $u \in E$ be arbitrary but fixed. We shall first consider a more simple evolution problem which is converted from (FDE:ϕ) by employing the above $u \in E$.

By fixing the functional term with $u \in E$, we consider a problem from (FDE:ϕ) by the type of

$$x'(t) + A(t, u_t)x(t) \ni G(t, u_t), \quad t \in [0, T], \quad x_0 = \phi.$$

For the simplicity, we put $B(t) \equiv A(t, u_t)$ and $g(t) \equiv G(t, u_t)$ for $t \in [0, T]$. Then our hypotheses (A.1)–(A.3) and the problem are converted as follows.

$$(\mathrm{EE} : \phi, u) \qquad x'(t) + B(t)x(t) \ni g(t), \quad t \in [0, T], \quad x_0 = \phi.$$

(B.1) For each $t \in [0, T]$, $B(t) : D(t) \subset X \to 2^X$ is m-accretive.
(B.2) For each $t, s \in [0, T]$ and $v \in X$,

$$\|B_\lambda(t)v - B_\lambda(s)v\| \leq L_0(\|v\|)|t - s|(1 + M)(1 + \|B_\lambda(s)v\|) \\ \equiv \tilde{L}_0(\|v\|)|t - s|(1 + \|B_\lambda(s)v\|)$$

where $\tilde{L}_0 : \mathcal{R}^+ \to \mathcal{R}^+$ is again nondecreasing continuous function with $\tilde{L}_0(p) = (1+M)L_0(p)$ and $B_\lambda(t)$ is the Yosida approximation of $B(t)$.
(B.3) For $t, s \in [0,T]$

$$\begin{aligned}\|g(t) - g(s)\| &\le \|G(t,u_t) - G(t,u_s)\| + \|G(t,u_s) - G(s,u_s)\| \\ &\le k_1\|u_t - u_s\|_C + L_1(\|u_s\|_C)|t-s| \\ &\le (k_1 M + L_1(\|u_s\|_C))|t-s| \\ &\le (k_1 M + L_1(\|\phi\|_C + MT))|t-s| \\ &\equiv \tilde{L}_1|t-s|\end{aligned}$$

where $\tilde{L}_1$ is a constant. Here we have used the below result $\|u_s\|_C \le \|\phi\|_C + MT$.

LEMMA 1. *Let* (A.1)–(A.4) *hold. Then, for fixed $u \in E$, there exist $C_i = C_i(\phi)$, $i = 1,2,3,4$, which are independent of u, such that*

$$\begin{aligned}|A(t,u_t)\phi(0)| = |B(t)\phi(0)| \le C_1 + C_2 T, \quad t \in [0,T], \\ \|G(t,u_t)\| = \|g(t)\| \le C_3 + C_4 T, \quad t \in [0,T]\end{aligned}$$

where

$$\begin{aligned}C_1 &= |A(0,\phi)\phi(0)|, \qquad C_2 = L_0(\|\phi(0)\|)(1 + M + C_1), \\ C_3 &= \|G(0,\phi)\|, \qquad C_4 = k_1 M + L_1(\|\phi\|_C).\end{aligned} \tag{1}$$

REMARK 1. We note that constants C_1–C_4 are dependent only on ϕ by (1).

Proof. First we show $\|u_t - \phi\|_C \le MT$. For $t \in [0,T]$ and $\theta \in [-r, 0]$, if $t + \theta > 0$, then

$$\begin{aligned}\|u_t(\theta) - \phi(\theta)\| &= \|u(t+\theta) - \phi(\theta)\| \\ &\le \|u(t+\theta) - \phi(0)\| + \|\phi(0) - \phi(\theta)\| \\ &\le k_0|\theta| + M|t+\theta| \le Mt \le MT.\end{aligned}$$

If $t + \theta \le 0$ then

$$\|u_t(\theta) - \phi(\theta)\| = \|\phi(t+\theta) - \phi(\theta)\| \le k_0 t \le MT.$$

Hence, $\|u_t - \phi\|_C = \sup_{\theta\in[-r,0]} \|u(t+\theta) - \phi(\theta)\| \le MT$.

By (A.2), we have

$$\begin{aligned}
&\|A_\lambda(t,u_t)\phi(0)\| \\
&\quad \le \|A_\lambda(0,\phi)\phi(0)\| + L_0(\|\phi(0)\|)\{|t-0|(1+\|A_\lambda(0,\phi)\phi(0)\|) \\
&\qquad\qquad + \|u_t-\phi\|_C\} \\
&\quad \le \|A_\lambda(0,\phi)\phi(0)\| + L_0(\|\phi(0)\|)\{T(1+\|A_\lambda(0,\phi)\phi(0)\|) + MT\}
\end{aligned}$$

for $t \in [0,T]$. Letting $\lambda \to 0$, we get

$$|A(t,u_t)\phi(0)| \le |A(0,\phi)\phi(0)| + TL_0(\|\phi(0)\|)\{1 + |A(0,\phi)\phi(0)| + M\}.$$

Therefore, $|A(t,u_t)\phi(0)| = |B(t)\phi(0)| \le C_1 + C_2T$.

Again by (A.3), for $t \in [0,T]$

$$\begin{aligned}
&\|G(t,u_t) - G(0,\phi)\| \\
&\quad \le \|G(t,u_t) - G(t,\phi)\| + \|G(t,\phi) - G(0,\phi)\| \\
&\quad \le k_1\|u_t - \phi\| + L_1(\|\phi\|_C)t \le k_1MT + L_1(\|\phi\|_C)T \\
&\quad = T(k_1M + L_1(\|\phi\|_C)).
\end{aligned}$$

It implies that for $t \in [0,T]$

$$\|g(t)\| = \|G(t,u_t)\| \le \|G(0,\phi)\| + T(k_1M + L_1(\|\phi\|_C)) = C_3 + C_4T. \quad \square$$

Let $\{t_j^n\}_{j=0}^n$ be a partition of the interval $[0,T]$ for fixed n, where $t_j^n = jh_n = jT/n$, $j = 0,1,\cdots,n$. And we let $g_j^n = g(t_j^n)$. When we put $x_0^n = \phi(0)$, we construct a sequence $\{x_j^n\}_{j=0}^n$ of elements of X satisfying

$$\frac{x_j^n - x_{j-1}^n}{h_n} + B(t_j^n)x_j^n \ni g_j^n, \qquad j = 1,2,\cdots,n$$

by m-accretiveness of B. The step function

$$x_n(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ x_j^n, & t \in (t_{j-1}^n, t_j^n],\ j = 1,2,\cdots,n \end{cases}$$

is called an approximate solution of (EE:ϕ,u). If the approximate solution converge to some continuous function uniformly on $[-r,T]$, we call it the limit solution of (EE:ϕ,u) on $[-r,T]$.

By the assumptions (B.1)–(B.3), we may conclude that conditions (A) and (C2) in Theorem 2 of Evans [4] are satisfied. So there exist a limit solution on $[-r,T]$ as in [4]. However, we calculate some bounds precisely to assure that they are independent of u.

LEMMA 2. *Let* (B.1)–(B.3) *and* (A.4) *hold. Then there exist constants* $C_5 = C_5(\phi)$ and $C_8 = C_8(\phi)$ *such that*

$$\sup_n\{\max_{0\le j\le n}\|x_j^n\|\}\le C_5, \text{ and } \quad \sup_n\{\max_{0\le j\le n}\frac{\|x_j^n - x_{j-1}^n\|}{h_n}\}\le C_8$$

where
(2)

$$\begin{aligned}
&C_5 = \|\phi(0)\| + (C_1 + C_3)T + (C_2 + C_4)T^2,\\
&C_6 = C_6(\phi) = L_0(\|\phi(0)\| + C_3 + (C_1 + C_3 + C_4)T + (C_2 + C_4)T^2),\\
&C_7 = C_7(\phi) = k_1 M + L_1(\|\phi\|_C + MT) + (1 + C_3 + C_4 T)C_6,\\
&C_8 = [(C_1 + C_3) + T(C_2 + C_4 + C_7)]\exp\{C_6 T\}.
\end{aligned}$$

Proof. We assume that n is sufficiently large so that $h_n < 1$ and $1 - h_n C_6 > 0$. And we set $g_j^n = g(t_j^n) = G(t_j^n, u_{t_j^n})$ and $J_\lambda^B(t) = J_\lambda(t, u_t) = (I+\lambda A(t, u_t))^{-1}$. Since $x_j^n = J_{h_n}^B(t_j^n)(x_{j-1}^n + h_n g_j^n)$,

$$\begin{aligned}
\|x_j^n - \phi(0)\| &= \|J_{h_n}^B(t_j^n)(x_{j-1}^n + h_n g_j^n) - J_{h_n}^B(t_j^n)\phi(0)\|\\
&\quad + \|J_{h_n}^B(t_j^n)\phi(0) - \phi(0)\|\\
&\le \|x_{j-1}^n - \phi(0)\| + h_n\|g_j^n\| + h_n\|B_{h_n}(t_j^n)\phi(0)\|\\
&\le \|x_{j-1}^n - \phi(0)\| + h_n(C_3 + C_4T) + h_n(C_1 + C_2T)\\
&\le \|x_{j-2}^n - \phi(0)\| + 2h_n(C_3 + C_4T) + 2h_n(C_1 + C_2T)\\
&\cdots\\
&\le \|x_0^n - \phi(0)\| + jh_n(C_3 + C_4T) + jh_n(C_1 + C_2T)\\
&= T\{(C_1 + C_3) + (C_2 + C_4)T\}
\end{aligned}$$

for $j = 1, 2, \cdots, n$. It implies that

$$\max_{1\le j\le n}\|x_j^n\| \le \|\phi(0)\| + (C_1 + C_3)T + (C_2 + C_4)T^2 = C_5.$$

Now we have a bound for $\|x_j^n - x_{j-1}^n\|/h_n$ with similar steps. In other words,

$$\begin{aligned}
\|x_j^n - x_{j-1}^n\| &= \|J_{h_n}^B(t_j^n)(x_{j-1}^n + h_n g_j^n) - J_{h_n}^B(t_{j-1}^n)(x_{j-2}^n + h_n g_{j-1}^n)\| \\
&\le \|J_{h_n}^B(t_j^n)(x_{j-1}^n + h_n g_j^n) - J_{h_n}^B(t_j^n)(x_{j-2}^n + h_n g_{j-1}^n)\| \\
&\quad + \|J_{h_n}^B(t_j^n)(x_{j-2}^n + h_n g_{j-1}^n) - J_{h_n}^B(t_{j-1}^n)(x_{j-2}^n + h_n g_{j-1}^n)\| \\
&\le \|x_{j-1}^n - x_{j-2}^n\| + h_n\|g_j^n - g_{j-1}^n\| \\
&\quad + h_n\|B_{h_n}(t_{j-1}^n)(x_{j-2}^n + h_n g_{j-1}^n) - B_{h_n}(t_j^n)(x_{j-2}^n + h_n g_{j-1}^n)\| \\
&\le \|x_{j-1}^n - x_{j-2}^n\| + h_n(k_1 M + L_1(\|\phi\|_C + MT))h_n \\
&\quad + h_n L_0(\|x_{j-2}^n\| + h_n\|g_{j-1}^n\|)|t_j^n - t_{j-1}^n| \\
&\quad \cdot (1 + \|B_{h_n}(t_{j-1}^n)(x_{j-2}^n + h_n g_{j-1}^n)\|.
\end{aligned}$$

Since $B_{h_n}(t_{j-1}^n)(x_{j-2}^n + h_n g_{j-1}^n) = g_{j-1}^n - (x_{j-1}^n - x_{j-2}^n)/h_n$ and $\|x_{j-2}^n\| \le C_5$,

$$\begin{aligned}
\|x_j^n - x_{j-1}^n\| &= \|x_{j-1}^n - x_{j-2}^n\| + h_n{}^2(k_1 M + L_1(\|\phi\|_C + MT)) \\
&+ h_n{}^2 L_0(C_5 + h_n(C_3 + C_4T))(1 + C_3 + C_4T + \|(x_{j-1}^n - x_{j-2}^n)/h_n\|.
\end{aligned}$$

It implies that

$$\begin{aligned}
&\max_{1\le k\le j} \|x_k^n - x_{k-1}^n\|/h_n \\
&\quad = \max_{1\le k\le j-1} \|x_k^n - x_{k-1}^n\|/h_n + h_n(k_1 M + L_1(\|\phi\|_C + MT)) \\
&\qquad + h_n(1 + C_3 + C_4T)L_0(C_5 + h_n(C_3 + C_4T)) \\
&\qquad + L_0(C_5 + h_n(C_3 + C_4T)) \max_{1\le k\le j} \|x_k^n - x_{k-1}^n\| \\
&\quad \le \max_{1\le k\le j-1} \|x_k^n - x_{k-1}^n\|/h_n + h_n(k_1 M + L_1(\|\phi\|_C + MT)) \\
&\qquad + h_n(1 + C_3 + C_4T)C_6 + C_6 h_n \max_{1\le k\le j} \|x_k^n - x_{k-1}^n\|/h_n
\end{aligned}$$

since $L_0(C_5 + h_n(C_3 + C_4T)) \le C_6(\phi) = C_6 = L_0(C_5 + C_3 + C_4T)$. Using $P_n = 1 - h_n C_6 \in (0, 1)$, we have

$$\frac{P_n}{h_n} \max_{1\le k\le j} \|x_k^n - x_{k-1}^n\| \le h_n C_7 + \frac{1}{h_n} \max_{1\le k\le j-1} \|x_k^n - x_{k-1}^n\|$$

where $C_7(\phi) = C_7 = k_1M + L_1(\|\phi\|_C + MT) + C_6(1 + C_3 + C_4T)$. Iterating this process, we get

$$\frac{P_n}{h_n}\max_{1\le k\le n}\|x_k^n - x_{k-1}^n\| \le h_nC_7 + \frac{h_nC_7}{P_n} + \frac{1}{P_nh_n}\max_{1\le k\le n-2}\|x_k^n - x_{k-1}^n\|$$

$$\le h_nC_7\sum_{s=0}^{n-2}\frac{1}{(P_n)^s} + \frac{1}{h_n(P_n)^{n-2}}\|x_1^n - x_0^n\|$$
$$\le h_nC_7\sum_{s=0}^{n-1}\frac{1}{(P_n)^s} + \frac{1}{h_n(P_n)^{n-1}}\|x_1^n - x_0^n\|.$$

Therefore, since $\|x_1^n - x_0^n\| \le h_n[(C_1 + C_3) + T(C_2 + C_4)]$,

$$\frac{1}{h_n}\max_{1\le k\le n}\|x_k^n - x_{k-1}^n\|$$
$$\le h_nC_7\sum_{s=1}^{n}\frac{1}{(P_n)^s} + \frac{1}{h_n(P_n)^n}\|x_1^n - x_0^n\|$$
$$\le h_nC_7\sum_{s=1}^{n}\frac{1}{(P_n)^s} + \frac{1}{(P_n)^n}((C_1 + C_3) + (C_2 + C_4)T).$$

Since

$$h_n\sum_{s=1}^{n}\frac{1}{(P_n)^s} \le h_n\sum_{s=1}^{n}\frac{1}{(P_n)^n} \le T/(1 - \frac{C_6T}{n})^n,$$

and $\lim_{n\to\infty}(1 - (C_6T)/n)^{-n} = \exp\{C_6T\}$,

$$\frac{1}{h_n}\max_{1\le k\le n}\|x_k^n - x_{k-1}^n\|$$
$$\le (C_7T + (C_1 + C_3) + (C_2 + C_4)T)\exp\{C_6T\}$$
$$\le ((C_1 + C_3) + (C_2 + C_4 + C_7)T)\exp\{C_6T\} = C_8.$$

Consequently,

$$\max_{1\le j\le n}\|\frac{x_j^n - x_{j-1}^n}{h_n}\| \le C_8. \quad \square$$

We now show that the constructed approximated solution $x_n(t)$ of (EE:ϕ, u) is a Lipshitz function so as to find Lipschitz constant of a limit solution $x_u(t)$ for (EE:ϕ, u).

LEMMA 3. *Let* (B.1)-(B.3) *and* (A.4) *hold. For sufficiently large* n, *there exists a constant* $C_9 = C_9(\phi)$ *such that*

$$\|x_n(t) - x_n(s)\| \le 2C_8T/n + C_9|t-s|, \qquad t,s \in [-r,T],$$

where $C_9 = \max\{k_0, C_8\}$ *which is independent of* n *and* u.

REMARK 2. Since a limit solution of (EE:ϕ, u) is the uniform convergence of an approximate solution $x_n(t)$, we may say that a limit solution is actually a Lipschitz continuous function with Lipschitz constant C_9. Most important things are C_9 is independent of u and a limit solution could be included in E if the interval T in C_9 is adjusted so that $C_9 \le M$.

Proof. We define a function

$$z_n(t) = \begin{cases} \phi(t), \ t \in [-r,0], \\ x_{j-1}^n + (t - t_{j-1}^n)\frac{x_j^n - x_{j-1}^n}{h_n}, \ t \in (t_{j-1}^n, t_j^n], \ j = 1, \cdots, n. \end{cases}$$

Then it is easy to show that $z_n(t)$ is a Lipschitz continuous with Lipschitz constant C_9. Moreover, since

$$\begin{aligned} \|x_n(t) - z_n(t)\| &\le \|x_j^n - x_{j-1}^n - (t - t_{j-1}^n)(x_j^n - x_{j-1}^n)/h_n\| \\ &\le \|(h_n - (t - t_{j-1}^n))(x_j^n - x_{j-1}^n)/h_n\| \\ &\le (t_j^n - t)\|(x_j^n - x_{j-1}^n)/h_n\| \le h_n C_8. \end{aligned}$$

for $t \in (t_{j-1}^n, t_j^n]$,

$$\begin{aligned} &\|x_n(t) - x_n(s)\| \\ &\qquad \le \|x_n(t) - z_n(t)\| + \|z_n(t) - z_n(s)\| + \|z_n(s) - x_n(s)\| \\ &\qquad \le 2h_n C_8 + C_9|t-s| \end{aligned}$$

for $t, s \in [-r,T]$. □

THEOREM 1. *Let* (A.1)-(A.4) *hold and* $\phi(0) \in \hat{D}$. *Then there exist a limit solution* $x_u(t)$ *of* (EE:ϕ,u) *on* $[-r,T]$ *for fixed* $u \in E$. *Moreover,* x_u *is Lipschitz continuous with Lipschitz constant* C_9 *on* $[-r,T]$.

Proof. By the assumption (B.1), $B(t)$ is m-accretive operator on X for $t \in [0,T]$. Thus, it satisfies the Condition (A) of Evans [4].

Also, since (B.2) and (B.3) imply the Conditions (C.2), we conclude that there exists a continuous function $x_u(t) : [-r, T] \to X$ which is the uniform convergence of the step function $x_n(t)$. Also, the limit solution x_u is Lipschitz continuous with constant C_9 by Lemma 3. □

Now we show the relation between the limit solutions of (EE:ϕ, u) and (EE:ϕ, v) for $u, v \in E$.

THEOREM 2. *Let $x_u(t)$ and $y_v(t)$ be the limit solutions of* (EE:ϕ, u) *and* (EE:ϕ, v) *in* Theorem 1, *respectively. Then for $0 \le \tau \le t \le T$*

$$\begin{aligned}\|x_u(t) - y_v(t)\| &\le \|x_u(\tau) - y_v(\tau)\| + C_6 T \|u - v\|_{[-r,T]} \\ &\quad + \int_\tau^t [x_u(\eta) - y_v(\eta), G(\eta, (x_u)_\eta) - G(\eta, (y_v)_\eta)]_+ \, d\eta.\end{aligned}$$

Proof. Let x_u, y_v be the limit solutions of (EE:ϕ, u), (EE:ϕ, v), respectively. By the definition of the limit solution of (EE:ϕ, u), there exists an approximate solution $x_n(t)$ such that

$$\frac{x_j^n - x_{j-1}^n}{h_n} + A(t_j^n, u_{t_j^n}) x_j^n \ni G(t_j^n, u_{t_j^n}), \tag{3}$$

$x_n(0) = x_0^n = \phi(0)$ and $x_n(t) = x_j^n$, $t \in (t_{j-1}^n, t_j^n]$, $j = 1, 2, \cdots, n$, where $h_n = t_j^n - t_{j-1}^n$. Also, there exists an approximate solution $y_m(t)$ such that

$$\frac{y_k^m - y_{k-1}^m}{\hat{h}_m} + A(s_k^m, v_{s_k^m}) y_k^m \ni G(s_k^m, v_{s_k^m}), \tag{4}$$

$y_m(0) = y_0^m = \phi(0)$ and $y_m(t) = y_k^m$, $t \in (s_{k-1}^m, s_k^m]$, $k = 1, 2, \cdots, m$, where $\hat{h}_m = s_k^m - s_{k-1}^m$. Let $\delta \in (0, T/2)$ and assume that n and m are sufficiently large such that $\max\{h_n, \hat{h}_m\} < \delta$. Then there is a positive constants $C_{10} = C_{10}(\phi)$ and $C_{11} = C_{11}(\phi)$ such that for $p \in \{0, 1, \ldots, n\}$ and $q \in \{0, 1, \ldots, m\}$

$$\begin{aligned}\|x_j^n - y_k^m\| &\le \|x_p^n - y_q^m\| + C_{11} D_{j,k} + \sum_{i=p}^{j} \delta_i^n h_n + \sum_{i=q}^{k} \hat{\delta}_i^m \hat{h}_m \\ &\quad + j h_n \{(\delta^{-1}\rho(T) + C_{10})(D_{j,k} + |t_p^n - s_q^m|) \\ &\quad + \rho(2\delta) + C_6(h_n + \|u - v\|_{[-r,T]})\}\end{aligned} \tag{5}$$

for $j = p, \cdots, n$ and $k = q, \cdots, m$ where

$$C_{10} = C_6(1 + C_3 + C_4T + C_8 + M), \quad \text{and}$$
$$C_{11} = \max\{C_{10}, 2C_3 + 2C_4T + C_8\}.$$

Here the symbols used above are defined by

$$\delta_j^n = \left[x_j^n - y_v(t_j^n), G(t_j^n, u_{t_j^n} - G(t_j^n, (y_v)_{t_j^n})\right]_\tau,$$

where $[x, y]_\tau = \tau^{-1}(\|x + \tau y\| - \|x\|)$ for $\tau > 0$,

$$\hat{\delta}_k^m = \|G(s_k^m, v_{s_k^m}) - G(s_k^m, (y_v)_{s_k^m})\| + \frac{2}{\tau}\|y_k^m - y_v(s_k^m)\|,$$
$$\rho(\hat{t}) = \sup\{\frac{2}{\tau}\|y_v(t) - y_v(r)\| + \|G(r, (y_v)_r) - G(t, (y_v)_t)\| : |t - r| \leq \hat{t}\}$$

and

$$D_{j,k} = \{((t_j^n - t_p^n) - (s_k^m - s_q^m))^2 + (t_j^n - t_p^n)h_n + (s_k^m - s_q^m)\hat{h}_m\}^{\frac{1}{2}}$$
$$+ \{((t_j^n - t_p^n) - (s_k^m - s_q^m))^2 + (t_j^n - t_p^n)h_n + (s_k^m - s_q^m)\hat{h}_m\}.$$

First, we prove that (5) holds. we let $\sigma = h_n\hat{h}_m/(h_n + \hat{h}_m)$. From (3) and (4), we have

$$A(t_j^n, u_{t_j^n})x_j^n \ni G(t_j^n, u_{t_j^n}) + \frac{x_{j-1}^n - x_j^n}{h_n},$$
$$A(s_k^m, v_{s_k^m})y_k^m \ni G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m}.$$

Choose $0 < \lambda < 1$. Then, with the similar steps in Lemma 5.1 of Evans [4],

$$J_{\sigma\lambda}(t_j^n, u_{t_j^n})\big(x_j^n + \sigma\lambda\big(G(t_j^n, u_{t_j^n}) + \frac{x_{j-1}^n - x_j^n}{h_n}\big)\big) = x_j^n,$$
$$J_{\sigma\lambda}(s_k^m, v_{s_k^m})\big(y_k^m + \sigma\lambda\big(G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m}\big)\big) = y_k^m.$$

From (A.2)-(A.4),

$$
\begin{aligned}
&\|x_j^n - y_k^m\| \\
&\le \|J_{\sigma\lambda}(t_j^n, u_{t_j^n})\big(x_j^n + \sigma\lambda(G(t_j^n, u_{t_j^n}) + \frac{x_{j-1}^n - x_j^n}{h_n})\big) \\
&\quad - J_{\sigma\lambda}(t_j^n, u_{t_j^n})\big(y_k^m + \sigma\lambda(G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\big)\| \\
&\quad + \|J_{\sigma\lambda}(t_j^n, u_{t_j^n})\big(y_k^m + \sigma\lambda(G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\big) \\
&\quad - J_{\sigma\lambda}(s_k^m, v_{s_k^m})\big(y_k^m + \sigma\lambda(G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\big)\| \\
&\le \|\big(x_j^n + \sigma\lambda(G(t_j^n, u_{t_j^n}) + \frac{x_{j-1}^n - x_j^n}{h_n})\big) \\
&\quad - \big(y_k^m + \sigma\lambda(G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\big)\| \\
&\quad + \sigma\lambda L_0(\|y_k^m + \sigma\lambda(G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\|)\{|t_j^n - s_k^m| \\
&\quad \cdot (1 + \|A_{\sigma\lambda}(s_k^m, v_{\hat{t}_k^m})(y_k^m + \sigma\lambda(G(s_k^m, v_{s_k^m} + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\|) \\
&\quad + \|u_{t_j^n} - v_{s_k^m}\|_C\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&x_j^n - y_k^m + \sigma\lambda\frac{x_{j-1}^n - x_j^n}{h_n} - \sigma\lambda\frac{y_{k-1}^m - y_k^m}{\hat{h}_m} \\
&= (1-\lambda)(x_j^n - y_k^m) + \frac{\lambda\hat{h}_m}{h_n + \hat{h}_m}(x_{j-1}^n - y_k^m) + \frac{\lambda h_n}{h_n + \hat{h}_m}(x_j^n - y_{k-1}^m),
\end{aligned}
$$

when we set $A_{j,k} = \|x_j^n - y_k^m\|$, we have

$$
\begin{aligned}
\lambda A_{j,k} + (1-\lambda)A_{j,k} = A_{j,k} &\le \frac{\lambda\hat{h}_m}{h_n + \hat{h}_m}A_{j-1,k} + \frac{\lambda h_n}{h_n + \hat{h}_m}A_{j,k-1} \\
&\quad + \|(1-\lambda)(x_j^n - y_k^m) + \sigma\lambda(G(t_j^n, u_{t_j^n}) - G(s_k^m, v_{s_k^m}))\| + U,
\end{aligned}
$$

where

$$U = \sigma\lambda L_0(\|y_k^m + \sigma\lambda(G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\|)\{|t_j^n - s_k^m| \\ \cdot (1 + \|A_{\sigma\lambda}(s_k^m, v_{t_k^m})(y_k^m + \sigma\lambda(G(s_k^m, v_{s_k^m}) \\ + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\|) + \|u_{t_j^n} - v_{s_k^m}\|_C\}.$$

It implies that

$$A_{j,k} \le \frac{\hat{h}_m}{h_n + \hat{h}_m} A_{j-1,k} + \frac{h_n}{h_n + \hat{h}_m} A_{j,k-1} + \frac{1-\lambda}{\lambda}(\|(x_j^n - y_k^m) \\ + \frac{\sigma\lambda}{1-\lambda}(G(t_j^n, u_{t_j^n}) - G(s_k^m, v_{s_k^m}))\| - \|x_j^n - y_k^m\|) + \frac{U}{\lambda} \\ = \frac{\hat{h}_m}{h_n + \hat{h}_m} A_{j-1,k} + \frac{h_n}{h_n + \hat{h}_m} A_{j,k-1} \\ + [x_j^n - y_k^m, \sigma(G(t_j^n, u_{t_j^n}) - G(s_k^m, v_{s_k^m}))]_\xi + \frac{U}{\lambda},$$

where $\xi = \lambda/(1-\lambda)$. By letting $\lambda \to 0$, since

$$\frac{U}{\lambda} \to \sigma L_0(\|y_k^m\|)\{|t_j^n - s_k^m|(1 + \|G(s_k^m, v_{s_k^m}) + \frac{y_{k-1}^m - y_k^m}{\hat{h}_m})\|) \\ + \|u_{t_j^n} - v_{s_k^m}\|_C\} \\ \le \sigma C_6\{|t_j^n - s_k^m|(1 + C_3 + C_4 T + C_8) + \|u_{t_j^n} - v_{s_k^m}\|_C\},$$

we have

$$\begin{aligned}
A_{j,k} \le{} & \frac{\hat{h}_m}{h_n+\hat{h}_m}A_{j-1,k} + \frac{h_n}{h_n+\hat{h}_m}A_{j,k-1} \\
& + [x_j^n - y_k^m, \sigma(G(t_j^n, u_{t_j^n}) - G(s_k^m, v_{s_k^m}))]_+ \\
& + \sigma C_6\{|t_j^n - s_k^m|(1 + C_3 + C_4T + C_8) + \|u_{t_j^n} - v_{s_k^m}\|_C\} \\
\le{} & \frac{\hat{h}_m}{h_n+\hat{h}_m}A_{j-1,k} + \frac{h_n}{h_n+\hat{h}_m}A_{j,k-1} + \sigma C_6\|u_{t_j^n} - v_{s_k^m}\|_C \\
& + \sigma\{C_6(1 + C_3 + C_4T + C_8)|t_j^n - s_k^m| \\
& + [x_j^n - y_k^m, \sigma(G(t_j^n, u_{t_j^n}) - G(s_k^m, v_{s_k^m}))]_+\} \\
\le{} & \frac{h_n}{h_n+\hat{h}_m}A_{j,k-1} + \frac{\hat{h}_m}{h_n+\hat{h}_m}A_{j-1,k} \\
& + \frac{h_n\hat{h}_m}{h_n+\hat{h}_m}\{C_6\|u_{t_j^n} - v_{s_k^m}\|_C + C_6(1 + C_3 + C_4T + C_8) \\
& \cdot |t_j^n - s_k^m| + \delta_j^n + \hat{\delta}_k^m + \rho(|t_j^n - s_k^m|)\}
\end{aligned}$$

by the fact that

$$\begin{aligned}
& [x_j^n - y_k^m, G(t_j^n, u_{t_j^n}) - G(s_k^m, v_{s_k^m})]_+ \\
& \le [x_j^n - y_v(t_j^n), G(t_j^n, u_{t_j^n}) - G(t_j^n, (y_v)_{t_j^n})]_\tau \\
& \quad + \|G(s_k^m, u_{s_k^m}) - G(s_k^m, (y_v)_{s_k^m})\| + \frac{2}{\tau}\|y_k^m - y_v(s_k^m)\| \\
& \quad + \|G(s_k^m, (y_v)_{s_k^m}) - G(t_j^n, (y_v)_{t_j^n})\| + \frac{2}{\tau}\|y_v(s_k^m) - y_v(t_j^n)\| \\
& \le \delta_j^n + \hat{\delta}_k^m + \rho(|t_j^n - s_k^m|).
\end{aligned}$$

Since

$$\begin{aligned}
|t_j^n - s_k^m| &\le |(t_j^n - s_k^m) - h_n| + h_n \\
&\le |(t_j^n - t_p^n) - (s_k^m - s_q^m) - h_n| + |t_p^n - s_q^m| + h_n \\
&\le D_{j-1,k} + |t_p^n - s_q^m| + h_n,
\end{aligned}$$

$$\begin{aligned}
\rho(|t_j^n - s_k^m|) &\le \delta^{-1}\rho(T)(|t_j^n - s_k^m|) - h_n) + \rho(2\delta) \\
&\le \delta^{-1}\rho(T)(D_{j-1,k} + |t_p^n - s_q^m|) + \rho(2\delta),
\end{aligned}$$

for some $p \in \{0, 1, \cdots, n\}$ and $q \in \{0, 1, \cdots, m\}$, and

$$\begin{aligned}\|u_{t_j^n} - v_{s_k^m}\|_C &\le \|u_{t_j^n} - u_{s_k^m}\|_C + \|u_{s_k^m} - v_{s_k^m}\|_C \\ &\le M|t_j^n - s_k^m| + \|u_{s_k^m} - v_{s_k^m}\|_C \\ &\le MD_{j-1,k} + M|t_p^n - s_q^m| + Mh_n + \|u_{s_k^m} - v_{s_k^m}\|_C,\end{aligned}$$

we have

$$\begin{aligned}A_{j,k} \le &\frac{h_n}{h_n + \hat{h}_m} A_{j,k-1} + \frac{\hat{h}_m}{h_n + \hat{h}_m} A_{j-1,k} \\ &+ \frac{h_n \hat{h}_m}{h_n + \hat{h}_m} \{(C_6(1 + C_3 + C_4T + C_8 + M) + \delta^{-1}\rho(T)) \\ &(D_{j-1,k} + |t_p^n - s_q^m|) + C_6(1 + C_3 + C_4T + C_8 + M)h_n \\ &+ \delta_j^n + \hat{\delta}_k^m + \rho(2\delta) + C_6\|u - v\|_{[-r,T]}\}.\end{aligned}$$

Consequently, when we put $C_{10} = C_6(1 + C_3 + C_4T + C_8 + M)$, we have

$$\begin{aligned}A_{j,k} \le &\frac{h_n}{h_n + \hat{h}_m} A_{j,k-1} + \frac{\hat{h}_m}{h_n + \hat{h}_m} A_{j-1,k} \\ &+ \frac{h_n \hat{h}_m}{h_n + \hat{h}_m} \{(C_{10} + \delta^{-1}\rho(T))(D_{j-1,k} + |t_p^n - \hat{t}_q^m|) \\ &+ C_{10}h_n + \delta_j^n + \hat{\delta}_k^m + \rho(2\delta) + C_6\|u - v\|_{[-r,T]}\}.\end{aligned} \tag{6}$$

At this moment, we consider $\|x_i^n - x_p^n\|$ for $i = p + 1, \cdots, n$. Since

$$\begin{aligned}|A(t_p^n, u_{t_p^n})x_p^n| &\le \|G(t_p^n, u_{t_p^n})\| + \|\frac{x_{p-1}^n - x_p^n}{h_n}\| \\ &\le C_3 + C_4T + C_8,\end{aligned}$$

by (A.2)

$$\begin{aligned}
&\|x_i^n - x_p^n\| \\
&\quad \le \|J_{h_n}(t_i^n, u_{t_i^n})(x_{i-1}^n + h_n G(t_i^n, u_{t_i^n}) - J_{h_n}(t_i^n, u_{t_i^n})x_p^n\| \\
&\qquad + \|J_{h_n}(t_i^n, u_{t_i^n})x_p^n - x_p^n\| \\
&\quad \le \|x_{i-1}^n - x_p^n\| + h_n\|G(t_i^n, u_{t_i^n})\| + h_n|A(t_i^n, u_{t_i^n})x_p^n| \\
&\quad \le \|x_{i-1}^n - x_p^n\| + h_n\|G(t_i^n, u_{t_i^n})\| + h_n|A(t_p^n, u_{t_p^n})x_p^n| \\
&\qquad + h_n L_0(\|x_p^n\|)\{|t_i^n - t_p^n|(1 + |A(t_p^n, u_{t_p^n})x_p^n|) + \|u_{t_i^n} - u_{t_p^n}\|_C\} \\
&\quad \le \|x_{i-1}^n - x_p^n\| + h_n(C_3 + C_4 T) + h_n(C_3 + C_4 T + C_8) \\
&\qquad + h_n C_6\{|t_i^n - t_p^n|(1 + C_3 + C_4 T + C_8) + M|t_i^n - t_p^n|\} \\
&\quad \le \|x_{i-1}^n - x_p^n\| + h_n C_{10}|t_i^n - t_p^n| + h_n(2C_3 + 2C_4 T + C_8) \\
&\quad \le \|x_{i-1}^n - x_p^n\| + h_n C_{11}|t_i^n - t_p^n| + h_n C_{11},
\end{aligned}$$

for $i = p+1, \cdots, n$ where $C_{11} = \max\{C_{10}, 2C_3 + 2C_4T + C_8\}$. If we add this inequality for $i = p+1, \cdots, j$, we have

$$\begin{aligned}
\|x_j^n - x_p^n\| &\le C_{11}h_n(j-p) + C_{11}h_n \sum_{i=p+1}^{j} |t_i^n - t_p^n| \\
&\le C_{11}h_n(j-p) + C_{11}(j-p)^2 {h_n}^2 \\
&= C_{11}|t_j^n - t_p^n| + C_{11}|t_j^n - t_p^n|^2 \\
&\le C_{11} D_{j,q}.
\end{aligned}$$

For $p \le j \le n$ and $k = q$,

$$\begin{aligned}
\|x_j^n - x_p^n\| &\le C_{11}(|t_j^n - t_p^n| + |t_j^n - t_p^n|^2) \\
&\le C_{11} D_{j,q},
\end{aligned}$$

which yields

$$\begin{aligned}
\|x_j^n - y_q^m\| &\le \|x_j^n - x_p^n\| + \|x_p^n - y_q^m\| \\
&\le \|x_p^n - y_q^m\| + C_{11} D_{j,q}.
\end{aligned}$$

Similarly, the above inequality also holds for $j = p$ and $q \le k \le m$. Next, let $p+1 \le j \le n$ and $q+1 \le k \le m$, and suppose that (5) holds for the pair $(j-1,k)$ and $(j,k-1)$. When we substitute (5) into (6), we get

$$
\begin{aligned}
A_{j,k} \le{}& \frac{h_n}{h_n+\hat h_m}\Big\{\|x_p^n - y_q^m\| + C_{11}D_{j,k-1} + \sum_{i=p}^{j}\delta_i^n h_n + \sum_{i=q}^{k-1}\hat\delta_i^m \hat h_m \\
&+ j h_n\big[(\delta^{-1}\rho(T)+C_{10})(D_{j,k-1}+|t_p^n - s_q^m|) + C_{10}h_n + \rho(2\delta) \\
&+ C_6\|u-v\|_{[-r,T]}\big]\Big\} \\
&+ \frac{\hat h_m}{h_n+\hat h_m}\Big\{\|x_p^n - y_q^m\| + C_{11}D_{j-1,k} + \sum_{i=p}^{j-1}\delta_i^n h_n + \sum_{i=q}^{k}\hat\delta_i^m \hat h_m \\
&+ (j-1)h_n\big[(\delta^{-1}\rho(T)+C_{10})(D_{j-1,k}+|t_p^n - s_q^m|) + C_{10}h_n \\
&+ \rho(2\delta) + C_6\|u-v\|_{[-r,T]}\big]\Big\} \\
&+ \frac{h_n\hat h_m}{h_n+\hat h_m}\Big\{(\delta^{-1}\rho(T)+C_{10})(D_{j-1,k}+|t_p^n - s_q^m|) + C_{10}h_n \\
&+ \rho(2\delta) + \delta_j^n + \hat\delta_k^m + C_6\|u-v\|_{[-r,T]}\Big\} \\
={}& \|x_p^n - y_q^m\| + C_{11}\Big(\frac{h_n}{h_n+\hat h_m}D_{j,k-1} + \frac{\hat h_m}{h_n+\hat h_m}D_{j-1,k}\Big) \\
&+ \sum_{i=p}^{j}\delta_i^n h_n + \sum_{i=q}^{k}\hat\delta_i^m \hat h_m + j h_n\big\{(\delta^{-1}\rho(T)+C_{10}) \\
&\cdot (D_{j,k}+|t_p^n - s_q^m|) + C_{10}h_n + \rho(2\delta) + C_6\|u-v\|_{[-r,T]}\big\} \\
\le{}& \|x_p^n - y_q^m\| + C_{11}D_{j,k} + \sum_{i=p}^{j}\delta_i^n h_n + \sum_{i=q}^{k}\hat\delta_i^m \hat h_m \\
&+ j h_n\big\{(\delta^{-1}\rho(T)+C_{10})(D_{j,k}+|t_p^n - s_q^m|) \\
&+ C_{10}h_n + \rho(2\delta) + C_6\|u-v\|_{[-r,T]}\big\}
\end{aligned}
$$

Here we have used

$$\frac{h_n}{h_n+\hat h_m}D_{j,k-1} + \frac{\hat h_m}{h_n+\hat h_m}D_{j-1,k} \le D_{j,k}.$$

Thus it turns out that (5) holds for the pair (j,k). Hence, we conclude that (5) holds for all $p \le j \le n$ and $q \le k \le m$. Let $\tau \in (t_{p-1}^n, t_p^n] \cap$

$(s_{q-1}^m, s_q^m]$ and $t \in (t_{j-1}^n, t_j^n] \cap (s_{k-1}^m, s_k^m]$. Letting $n, m \to \infty$ in (5), (7)

$$\|x_u(t) - y_v(t)\| \le \|x_u(\tau) - y_v(\tau)\| + \limsup_{n\to\infty} \sum_{i=p}^{j} \delta_i^n h_n + \limsup_{m\to\infty} \sum_{i=q}^{k} \hat{\delta}_i^m \hat{h}_m + T\rho(2\delta) + C_6 T\|u - v\|_{[-r,T]}.$$

Since

$$\lim_{n\to\infty} \sum_{i=p}^{j} \delta_i^n h_n = \int_\tau^t [x_u(\eta) - y_v(\eta), G(\eta, (x_u)_\eta) - G(\eta, (y_v)_\eta)]_\tau d\eta$$

and $\lim_{m\to\infty} \sum_{i=q}^{k} \hat{\delta}_i^m \hat{h}_m = 0$, letting $\delta \downarrow 0$ in (7)

$$\|x_u(t) - y_v(t)\| \le \|x_u(\tau) - y_v(\tau)\| + C_6 T\|u - v\|_{[-r,T]} + \int_\tau^t [x_u(\eta) - y_v(\eta), G(\eta, (x_u)_\eta) - G(\eta, (y_v)_\eta)]_\tau d\eta$$

Again, by letting $\tau \downarrow 0$ for the above inequality, we finally have desired result. □

THEOREM 3. *Let $\phi(0) \in \hat{D}$ and (A.1)–(A.4) hold. Then there exists $\bar{T} \in (0, T]$ such that (FDE:ϕ) has a unique generalized solution on $[0, \bar{T}]$.*

Proof. Let $u, v \in E$ be arbitrary. By Theorem 2, for $t \in [0, T]$ we have

$$\begin{aligned}
&\|x_u(t) - y_v(t)\| \\
&\le C_6 T\|u - v\|_{[-r,T]} + \int_0^t \|G(\eta, (x_u)_\eta) - G(\eta, (y_v)_\eta)\|\, d\eta \\
&\le C_6 T\|u - v\|_{[-r,T]} + \int_0^t k_1\|x_u - y_v\|_{[-r,T]}\, d\eta \\
&\le C_6 T\|u - v\|_{[-r,T]} + k_1 T\|x_u - y_v\|_{[-r,T]}.
\end{aligned}$$

Therefore,

$$(8) \qquad \|x_u - y_v\|_{[-r,T]} \le C_6 T\|u - v\|_{[-r,T]} + k_1 T\|x_u - y_v\|_{[-r,T]}.$$

for $u, v \in E$. Noting that $C_6 = L_0(\|\phi(0)\| + C_3 + (C_1 + C_3 + C_4)T + (C_2 + C_4)T^2)$ is independent of u, v, we set

$$(9) \quad T_1 = \frac{-(C_1 + C_3 + C_4) + \sqrt{(C_1 + C_3 + C_4)^2 + 4(C_2 + C_4)}}{2(C_2 + C_4)},$$

$$(10) \quad T_2 = 1/(k_1 + K_1 + M), \quad \text{where } K_1 = L_0(\|\phi(0)\| + C_3 + 1),$$

$$(11) \qquad T_3 = \frac{M - (C_1 + C_3)e}{(C_2 + C_4 + K_2)e},$$

where $K_2 = k_1 M + L_1(\|\phi\|_C + 1) + (2 + C_3)K_1$. Let $\bar{T} = \min\{T, T_1, T_2, T_3\}$. Then, for the interval $[-r, \bar{T}]$, we have same result as in Theorem 2. In other words,

$$\|x_u - y_v\|_{[-r,\bar{T}]} \le C_6 \bar{T} \|u - v\|_{[-r,\bar{T}]} + k_1 \bar{T} \|x_u - y_v\|_{[-r,\bar{T}]}.$$

But $(C_1 + C_3 + C_4)\bar{T} + (C_2 + C_4)\bar{T}^2 < 1$ by (9). Moreover $C_6 < L_0(\|\phi(0)\| + C_3 + 1) = K_1$. It implies that $\exp\{C_6 \bar{T}\} < \exp\{K_1 \bar{T}\} < e$ by (10) and $C_7 < K_2$ by (2). Therefore, on $[-r, \bar{T}]$,

$$(12) \qquad \|x_u - y_v\|_{[-r,\bar{T}]} \le K_1 \bar{T} \|u - v\|_{[-r,\bar{T}]} + k_1 \bar{T} \|x_u - y_v\|_{[-r,\bar{T}]}.$$

We replace T by $\bar{T}$ in the set E. Since

$$\begin{aligned} C_8 &= [(C_1 + C_3) + \bar{T}(C_2 + C_4 + C_7)] \exp\{C_6 \bar{T}\} \\ &\le [(C_1 + C_3) + \frac{M - (C_1 + C_3)e}{(C_2 + C_4 + K_2)e}(C_2 + C_4 + C_7)]e \\ &< (C_1 + C_3)e + M - (C_1 + C_3)e = M, \end{aligned}$$

we may conclude that $C_9 = \max\{k_0, C_8\} < M$. By Lemma 2, the limit solution x_u is included in E for confined interval $[-r, \bar{T}]$ for $u \in E$. Therefore, $x_u \in E$ for all $u \in E$. If we define an operator $F : E \to E$ by $u \mapsto x_u$, where $x_u(t)$ is the limit solution of (EE:ϕ, u), then F is a strict contraction on a complete metric space E by (10) and (12). By the Banach fixed point theorem, there is a unique fixed point of F in E, say $x(t)$ for $t \in [-r, \bar{T}]$. Then, $x(t)$ is the unique generalized solution of (FDE:ϕ) which is Lipschitz continuous on $[-r, \bar{T}]$. □

REMARK 3. It is obvious from the proof of the above theorems that the interval $[0,T]$ can be replaced by $[\bar{T},T]$. Then the solution $x(t)$ of (FDE:ϕ) exists beyond $\bar{T}$. With this processing, we may conclude that there exists a maximal interval of existence of solutions of (FDE:ϕ) on $[0,T]$.

REMARK 4. Using the result of Theorem 2, we may have similar result of Ha, Shin and Jin [6] with the concept of integral solution defined by Benilan. It is quite interested in investigating the relation between two evolution operators generated by operators in (FDE:ϕ) with different second terms. Also, for a just continuous perturbation $G(t,\cdot)$, we may apply the method in the paper of Kartsatos and Shin [11].

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