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ON THE CRITICAL RIEMANNIAN METRIC

Won Tae Oh

1. Introduction

It is well known that the symplectic manifold is a C^∞ manifold M^{2n} together with closed 2-form Ω such that $\Omega^n \neq 0$. Kaehler Manifold and cotangent bundles are well known examples. In 1969, S.I. Goldberg conjuctured that a compact almost-Kaehler, Einstein manifold is Kaehlerian. Thurston gave an example of compact symplectic manifold with no Kaehler structure for topological reason in 1976. E. Abbena gave a natural almost Kaehler metric on this manifold and computed its curvature. After then a family of compact homogeneous manifolds M^{2n+2} admitting almost Kaehler structures which are not Kaehlerian, has been reported in [2]. Those manifolds were the analoges of Abbena's case.

On the other hand, the critical points of the function $I(g) = \int_M R\, dV_g$, where R is the scalar structure of the metric g, defined on the set of all Riemannian metrics of the same total volume on a compact manifold M are Einstein metrics[4]. D. E. Blair and Ianus[3] gave a necessary and sufficient conditions of the function defined on the set of metrics associated to a symplectic form on a compact symplectic manifold has the critical point of $K(g) = \int R - R^* dV$, where R^* is the $*$ - scalar curvature.

The main purpose of this paper is to show that the Abbena metric is a critical point of the given intgral function. In section 2, we introduce the 4-dimensional symplectic manifold which are not Kaehlerian. In section 3, we study a integral function with the critical point as Abbena metric.

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2. 4-dimensional symplectic manifold with no Kaehler structure

In [1] E. Abbena introduce a 4-dimensional compact homogeneous space $M = G/\Gamma$ where G is a certain connected Lie group and Γ a discrete subgroup. This manifold was defined by W. Thurston as a fiber bundle over the 2-torus and was known as an example of a compact symplectic manifold with no Kaehler structure. He shows that the first betti number of this manifold is 3 whereas the odd-dimensional Betti numbers of a compact Kaehler are even. E. Abbena defined a metric on this manifold and a compatible almost complex structure on M. From now on we call this manifold as Abbena-Thurston manifolds.

In this section we will introduce the Abbena-Thurston manifold and its curvature. Let G be the closed connected subgroup of $GL(4, C)$ defined by

$$\left\{ \left(\begin{array}{cccc} 1 & a_{12} & a_{13} & 0 \\ 0 & 1 & a_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi i a} \end{array} \right) \middle| \; a_{12},\ a_{13},\ a_{23},\ a \in R \right\}.$$

Then G is the product of the Heisenberg group H and S^1. Let Γ be the discrete subgroup of G with integer entries and $M = G/\Gamma$. Denote by $x,\ y,\ z,\ t$ coordinates on G, say for $A \in G$, $x(A) = a_{12}$, $y(A) = a_{23}$, $z(A) = a_{13}$, $t(a) = a$. If L_B is the left translation by $B \in G$, then $L_B^* dx = dx$, $L_B^* dy = dy$, $L_B^*(dz - xdy) = dz - xdy$, and $L_B^* dt = dt$. In particular these forms are invariant under the action of Γ ; let $\pi : G \to M$, then there exist 1-forms $\alpha_1,\ \alpha_2,\ \alpha_3$, and α_4 on M such that $dx = \pi^*\alpha_1$, $dy = \pi^*\alpha_2$, $dz - xdy = \pi^*\alpha_3$, and $dt = \pi^*\alpha_4$. Setting $\Omega = \alpha_4 \wedge \alpha_2 + \alpha_2 \wedge \alpha_3$, we see that $\Omega \wedge \Omega \neq 0$ and $d\Omega = 0$ on M. Hence M admits a symplectic structure. The vector fields $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$, $e_3 = \frac{\partial}{\partial z}$, $e_4 = \frac{\partial}{\partial t}$ are dual to $dx,\ dy,\ dz - xdy,\ dt$ respectively and are left invariant. Moreover $\{e_i\}$ is orthonormal with respect to the left invariant metric on G given by $ds^2 = dx^2 + dy^2 + (dz - xdy)^2 + dt^2$. On M the corresponding metric is $g = \sum \alpha_i \otimes \alpha_i$. The Riemannian manifold (M, g) is refered to as the Abbena-Thurston manifold. Moreover M carries an almost complex structure defined by $Je_1 = e_4$, $Je_2 = -e_3$, $Je_3 = e_2$, $Je_4 = -e_1$. Then noting that $\Omega(X,\ Y) = g(X,\ JY)$, we see that g is an associated metric. With respect to the basis $\{e_i\}$ the components R_{kjih} of the curvature tensor are all zero except $R_{1221} = \frac{3}{4}$, $R_{2332} = -\frac{1}{4}$, and $R_{1331} = -\frac{1}{4}$. Thus the Ricci tensor Q is given by the matrix

$$\begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and we note that this manifold is not Einstein but has the constant scalar curvature.

3. Abbena metric as a Crtitical point of Integral function

Let M be a compact orientable manifold and $\mathcal{M}$ the set of all Riemannian metrics on M having the same total volume. Then we have a lemma[3 of p.25]

LEMMA. *Let T be a second order symmectric tensor field on M. Then $\int_M T^{ij} D_{ij} dV_g = 0$ for all symmetric tensor fields D satisfying $\int_M D^i_i dV_g = 0$ if and only if $T = cg$ for some constant c.*

Now we consider the Riemannian geometry of symplectic manifold. For a symplectic manifold M let k be any Riemannian metric and X_1, ..., X_{2n} be a k-orthonormal basis. Consider the $2n \times 2n$ matrix $\Omega(X_i, X_j)$; it is non-singular and hence may be written as the product GF of a positive definite symmetric matrix G and an orthogonal matrix F. Then G defines a new metric g and F defines an almost complex structure J; checking the overlaps of local charts, it is easy to see that g and J are globally defined on M. The key point metric g created in this way is called associated metrics. They have the same volume element $dV = \frac{1}{2^n n!}\Omega^n$. Now we let $\mathcal{A}$ the set of all associated metrics on M and define

$$A[g] = \int_M R_{kjih} R^{kjih} dV_g. \tag{1}$$

on $\mathcal{A}$. A C^∞ curve on $\mathcal{A}$ will be represented locally by $g_{ji}(x_1, x_2, ..., x_n; t)$ and we define a tensor field D_{ji} on $(M, g(t))$ by $D_{ji}(x,t) = \frac{\partial g_{ji}(x,t)}{\partial t}$. This symmetric $(0, 2)$-tensor satisfies $\int_M D_p{}^p dV = 0$. The curvatue tensor ${R_{kji}}^h$ changes with g and we get $\frac{\partial}{\partial t}{R_{kji}}^h = \partial_k {D_{ji}}^h - \partial_j {D_{ki}}^h$, where the tensor ${D_{ji}}^h$ is defined by ${D_{ji}}^h = \{{}^{\,h}_{ji}\}$. It satisfies ${D_{ji}}^h = \frac{1}{2}(\nabla_j {D_i}^h + \nabla_i {D_j}^h - \nabla^h D_{ji})$ and ∇ means the covariant differentiation with respect to the metric tensor $g(t)$. As we have $\frac{\partial}{\partial t} g_{ih} = -g^{ki} g^{jh} \frac{\partial}{\partial t} g_{kj} = -D^{ih}$, we get $\frac{\partial}{\partial t}(R_{kjih} R^{kjih}) = 4{R^{kji}}_h \nabla_k \nabla_i {D_j}^h - 2{R_{kji}}^b {R^{kji}}_a D^a_b$. Now, from (1) we get

$$\frac{d}{dt} A[g(t)] = \int_M [\frac{\partial}{\partial t}(R_{kjih} R^{kjih}) + \frac{1}{2} R_{kjih} R^{kjih} g^{qp} D_{qp}] dV.$$

Applying Green's theorem, we get

$$\frac{d}{dt} A[g(t)] = \int_M [4(\nabla_i \nabla_k {R^{kji}}_h) D^h_j - 2{R_{kji}}^q R^{kjip} D_{qp} + \frac{1}{2} R_{kjih} R^{kjih} {D_p}^p] dV.$$

With the help of Ricci's identity and Bianchi's identity, we have

$$\frac{d}{dt} A[g(t)] = \int_M [2\nabla^j \nabla^i R - 4\nabla_p \nabla^p R^{ji} + 4{R^j}_p R^{pi}$$

$$-4{{R^j}_{qp}}^i R^{qp} - 2R^{srqj} {R_{srq}}^i + \frac{1}{2} R_{dcba} R^{dcba} g^{ji}] D_{ji} dV.$$

From the Lemma, we see that g is a critical point of $A[g]$ if and only if $2\nabla^j \nabla^i R - 4\nabla_p \nabla^p R^{ji} + 4{R^j}_p R^{pi} - 4{{R^j}_{qp}}^i R^{qp} - 2R^{srqj} {R_{srq}}^i + \frac{1}{2} R_{dcba} R^{dcba} g^{ji} = c g^{ji}$ for some constant c. By transvecting with g_{ji}, we get $c = -\frac{2}{n} \nabla_p \nabla^p R + (\frac{1}{2} - \frac{2}{n}) R_{dcba} R^{dcba}$. Considering the curvature of Abbena-Thurston manifold stated in **2**, we have the following theorem.

THEOREM. *Let M be a Abbena-Thurston manifold and $\mathcal{A}$ be the set of all associated metrics on M . Then the Abbena metric g is a critical point of the functional $A[g]$ defined on $\mathcal{A}$.*

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Department of Mathematics
Chungbuk National University
Cheongju 360-763, Korea