Pusan Kyŏngnam Math. J. 12(1996), No. 1, pp. 107–117

k–INVARIANT HYPERSURFACE OF $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$

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0. Introduction

Yano[1] studied the differential geometry of $S^n \times S^n$ and introduced the structure equations of real hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

S.-S.Eum, U-H.Ki and Y.H.Kim [2] researched partially real hypersurfaces of $S^n \times S^n$ by using the concept of k-invariance.

In [3], the author found that the necessary and sufficient condition for a hypersurface of $S^n \times S^n$ being k-antiholomorphic and investigated its global properties

The purpose of the present paper is devoted to study the necessary and sufficient condition for real hypersurfaces of $S^n \times S^n$ being k-invariant, and characterize their global properties.

In section 1, we recall the structure equations of hypersurfaces of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

In section 2, we find the necessary and sufficient condition for a hypersurface of $S^n \times S^n$ being k-invariant, and prove that it is isometric to $S^{n-1} \times S^n$.

1. Structure equations of hypersurfaces of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$

Let M be a hypersurface immersed isometrically in $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ as a submanifold of codimension 2 of $(2n+2)$-dimensional Euclidean space or real hypersurface of $(2n+1)$-dimensional unit sphere $S^{2n+1}(1)$. And we suppose that M is covered by the system of coordinate neighborhoods $\{\bar{V}; \bar{x}^a\}$, where here and in the sequel, the indices $a, b, c, d, \cdots$ run over the range $\{1, 2, \cdots, 2n-1\}$.

Since the immersion $i : M \to S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ is isometric, from the (f, g, u, v, λ)-structure defined on $S^n \times S^n$, we get the so-called $(f, g, u, v, w, \lambda, \mu, \nu)$-structure [2] given by

Received May 10,1996.

$$f_b^e f_e^a = -\delta_b^a + u_b u^a + v_b v^a + w_b w^a, \tag{1.1}$$

$$\begin{aligned} f_e^a u^e &= -\lambda v^a + \mu w^a, \\ f_e^a v^e &= \lambda u^a + \nu w^a, \\ f_e^a w^e &= -\mu u^a - \nu v^a \end{aligned} \tag{1.2}$$

or, equivalently

$$\begin{aligned} &u_e f_a^e = \lambda v_a - \mu w_a, \quad v_e f_a^e = -\lambda u_a - \nu w_a, \quad w_e f_a^e = \mu u_a + \nu v_a, \\ &u_e u^e = 1 - \lambda^2 - \mu^2, \quad u_e v^e = -\mu\nu, \quad u_e w^e = -\lambda\nu, \\ &v_e v^e = 1 - \lambda^2 - \nu^2, \quad v_e w^e = \lambda\mu, \\ &w_e w^e = 1 - \mu^2 - \nu^2 \end{aligned}$$

where u_a, v_a and w_a are 1-forms associated with u^a, v^a and w^a respectively given by $u_a = u^b g_{ba}$, $v_a = v^b g_{ba}$ and $w_a = w^b g_{ba}$, and $f_{ba} = f_b^c g_{ca}$ is skew-symmetric. Moreover, we obtain

$$k_e^e = -\alpha, \tag{1.3}$$

$$k_c^e k_e^a = \delta_c^a - k_c k^a, \tag{1.4}$$

$$k_c^e f_e^a + f_c^e k_e^a = k_c w^a - w_c k^a, \tag{1.5}$$

$$k_c^e u_e = -v_c - \mu k_c, \quad k_c^e v_e = -u_c - \nu k_c, \tag{1.6}$$

$$\nabla_d l_{cb} - \nabla_c l_{db} = k_d k_{cb} - k_c k_{db}, \tag{1.7}$$

$$\nabla_c u_b = \mu l_{cb} - \lambda k_{cb} + f_{cb}, \tag{1.8}$$

$$\nabla_c w_b = -\mu g_{cb} - \nu k_{cb} + k_c v_b - l_{ce} f_b^e, \tag{1.9}$$

$$\nabla_c \lambda = -2v_c, \nabla_c \mu = w_c - \lambda k_c - l_{ce} u^e, \nabla_c \nu = k_{ce} w^e - l_{ce} v^e, \tag{1.10}$$

$$\nabla_c k_b^a = l_{cb} k^a + l_c^a k_b, \tag{1.11}$$

$$\nabla_c k_b = -k_{ba} l_c^a + \alpha l_{cb}, \tag{1.12}$$

$$\nabla_c \alpha = -2 l_{ce} k^e, \tag{1.13}$$

From these structure equations, we can easily see that the 1-form k_c is the third fundamental tensor when M is considered as a submanifold of codimension 2 immersed in $S^{2n+1}(1)$.

Finally, we introduce the followings

REMARK[4]. *If $\lambda^2 + \mu^2 + \nu^2 = 1$ on the hypersurface M, we see that*

$$\mu = 0, \nu = constant(\neq 0), v_c = 0 \quad \text{and} \quad \alpha = 0.$$

And if the function λ vanishes on some open set, then we have $v_c = 0$ and $\mu = 0$. Moreover the 1-form u_b never vanishes on an open set in M, in fact, if the 1-form u_b is zero on an open set in M, then $f_{cb} = 0$, which contradict $n > 1$.

LEMMA 1.1 [3]. *Let M be a hypersurface satisfying $k_{ce} f_b^e = k_{be} f_c^e$ of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$. Then we have*

$$\lambda^2 + \mu^2 + \nu^2 = 1 \quad or \quad \mu^2 + \nu^2 + \alpha\mu\nu = 0$$

on M

2. k-Invariant hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$

Let M be a hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ such that $k_c^e f_e^a + f_c^e f_e^a = 0$ holds every point of M or, equivalently

$$k_{ce} f_b^e = k_{be} f_c^e. \tag{2.1}$$

Then we have (2.2) – (2.6) (see [3]),

$$(1-\mu^2-\nu^2)k_c = \theta w_c, \quad (1-\alpha^2)w_c = \theta k_c, \tag{2.2}$$

$$k_{ce} w^e = -\alpha w_c, \tag{2.3}$$

$$(\mu^2+\nu^2)k_c + (\mu+\alpha\nu)v_c + (\nu+\alpha\mu)u_c = 0, \tag{2.4}$$

$$(\mu^2+\nu^2)(1-\alpha^2)(1-\lambda^2-\mu^2-\nu^2) = 0, \tag{2.5}$$

$$(\nu+\alpha\mu)^2 + (\mu+\alpha\nu)^2 = \mu^2+\nu^2+2\alpha\mu\nu. \tag{2.6}$$

First of all we prove

LEMMA 2.1. *Let M be a hypersurfave with (2.1) of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$. If the function α is constant on M, then M is k-invariant or k-antiholomorphics.*

Proof. Since α is constant on M, (1.13) gives

$$l_{ce} k^e = 0.$$

Hence, the second relationship of (2.2) means that

$$(1-\alpha^2) l_{ce} w^e = 0. \tag{2.7}$$

We now suppose that

$$(1-\alpha^2)(1-\lambda^2-\mu^2-\nu^2) \neq 0$$

at some point p of M. Then (2.5) implies that

$$\mu(p) = \nu(p) = 0. \tag{2.8}$$

So the second equation of (1.10) gives

$$l_{ce}u^e = (1 - \theta\lambda)w_c \tag{2.9}$$

at the point p because of (2.2) with $\mu = \nu = 0$.

From (2.8) and the fact that

$$w_e w^e = 1 - \mu^2 - \nu^2,$$

we have $w_e w^e = 1$ at $p \in M$. So the second equation of (2.2) means that

$$\alpha^2 + \theta^2 = 1$$

at the point p. Consequently the function θ is non-zero covariant constant at $p \in M$.

Transvecting (2.9) with w^c and taking account of (2.8) and the fact that $w_e w^e = 1$ at $p \in M$, we get at the point because of $(1 - \alpha^2)(p) \neq 0$. The constant θ being nonzero, λ is constant at $p \in M$. Therefore, the first equation of gives $v_c = 0$ at the point p. So

$$v_e v^e = 1 - \lambda^2 - \nu^2$$

leads to $(1 - \lambda^2)(p) = 0$ and hence $u_c = 0$ at the point p because of (2.8). According to Remark, it is contradictory.

Thus, it follows that

$$(1 - \alpha^2)(1 - \lambda^2 - \mu^2 - \nu^2) = 0 \tag{2.10}$$

on M.

If $1 - \lambda^2 - \mu^2 - \nu^2 = 0$ on M, then α vanishes identically because of Remark in Section 1. Since α is constant, we see that M is k-invariant or k-antiholomorphic. Thus Lemma 2.1 is proved.

LEMMA 2.2. *Let M be a hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$. In order that the hypersurface is k-invariant, it is necessary and sufficient that*

$$k_c^e f_e^a + f_c^e k_e^a = 0, \quad l_c^e k_e^a - k_c^e l_e^a = 0 \tag{2.11}$$

hold on M.

Proof. The sufficiency comes immediately from (1.5) and (1.12). Conversely, suppose that (2.11) is satisfied on M, then from Lemma 1.1 we have

$$\lambda^2 + \mu^2 + \nu^2 = 1 \quad \text{or} \quad \mu^2 + \nu^2 + 2\alpha\mu\nu = 0$$

on M. If we assume that the first equation of (2.12) holds on M,then

$$v_c = 0, \quad \nu = constant(\neq 0)$$

because of Remark. So the first equation of (1.6) gives

$$u_c + \nu k_c = 0.$$

Differentiating this covariantly and substituting (1.8) and (1.12), we find

$$u l_{cb} - \lambda k_{cb} + f_{cb} = \nu(\alpha l_{cb} - k_{be} l_c^e)$$

because ν is constant, from which, taking the skew-symmetric part,

$$2f_{cb} = \nu(k_{ce} l_b^e - k_{be} l_c^e).$$

Thus, it contradicts the fact that the tensor f_c^a has maximal rank and the second relationship of (2.11).

Therefore, we obtain from (5.12) that

$$\mu^2 + \nu^2 + 2\alpha\mu\nu = 0$$

on M. In this case, we have $\nu + \alpha\mu = 0$, $\mu + \alpha\nu = 0$ on M because of (2.6), and hence $\mu^2 = \nu^2$.

Consequently, (2.4) gives

$$\mu k_c = 0. \tag{2.13}$$

If the hypersurface is not k-invariant, then we have $\mu = 0$ and $\nu = 0$.

Hence (1.10) implies

$$l_{ce}u^e = (1 - \theta\lambda)w_c, \quad l_{ce}v^e = -\alpha w_c \tag{2.14}$$

where we have used (2.2) with $\mu = \nu = 0$ and (2.3).

Transvecting the second equation of (2.11) with $v^c w^b$ and $u^b w^b$ successively and taking account of (1.6), (2.3) and (2.14), we find respectively

$$\theta\lambda = 2, \alpha^2 = -1 + \theta\lambda$$

because of the fact that $\mu = \nu = 0$, which implies

$$1 - \alpha^2 = 0, \quad \text{i.e.,} \quad k_c = 0.$$

Thus we see from (2.13) that the hypersurface is k-invariant. This completes the proof of Lemma 2.2.

THEOREM 2.3. *If M be is a k-invarinat hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2}) (n > 1)$ satisfying*

$$l_c^e f_e^a + f_c^e l_e^a = 0, \tag{2.15}$$

then M is totally geodesic. Moreover, M is complete and M is $S^{n-1} \times S^n$.

Proof. Since M is k-invariant, that is $k_c = 0$, (1.12) reduces to

$$k_{be} l_c^e = \alpha l_{cb} \tag{2.16}$$

Transvecting this with w^b and making use of (2.3) and the fact that $1 - \alpha^2 = 0$, we get

$$l_{be}w^e = 0, \tag{2.17}$$

where we have used the result of Lemma 2.2.

Differentiating the last expression covariantly and substituting with $k_c = 0$, we get

$$(\nabla_c l_{be})w^e + l_b^e(-\mu g_{ce} - \nu k_{ce} - l_{ca} f_e^a) = 0,$$

from which, taking the skew-summmetric part and considering (1.7) with $k_c = 0$ and the second equation of (2.11),

$$l_b^e l_{ca} f_e^a = 0,$$

or, using (2.15)

$$l_b^e l_{ea} f_c^a = 0.$$

If we transvect with f^{cb}, we obtain $\|l_{ce} f_b^e\|^2 = 0$ and hence

$$l_{ce} f_b^e = 0, \tag{2.18}$$

which together with (1.1) yields

$$l_{ce}(-\delta_b^e + u_b u^e + v_b v^e) = 0 \tag{2.19}$$

because of (2.17).

Applying v^b to (2.18) and making use of (1.2) and (2.17), we find

$$\lambda l_{ce} u^e = 0.$$

Since the hypersurface is k-invarinat, remembering Remark, the function λ does not vanish. Thus,

$$l_{ce} u^e = 0. \tag{2.20}$$

Transvecting (2.16) with v^b, gives

$$l_{ce} v^e = 0. \tag{2.21}$$

because of (1.6) with $k_c = 0$ and (2.20). Using (2.20) and (2.21), the equation (2.19) reduces $l_{cb} = 0$, which shows that the hypersurface is totally geodesic.

Since we have $1 - \alpha^2 = 0$ on M, (1.3),(1.4) and (1.11) reduce respectively to

$$k_e^e = \pm 1, \quad k_c^e k_e^a = \delta_c^a, \quad \nabla_c k_b^a = 0. \tag{2.22}$$

Since the hypersurface M is totally geodesic, the second fundamental tensor k_c^a of M in $S^{2n+1}(1)$ has the form

$$(k_c^a) = \begin{pmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & -1 & & \\ & 0 & & \ddots & \\ & & & & -1 \end{pmatrix}$$

for a suitable orthonomal frame. The first relationship of (2.22) means that the multiplicity of the eigenvalue 1 of k_c^a is $n-1$ or n. Now, we consider two distributions on M

$$D_1 = \{X \mid kX = X\}, \quad D_2 = \{X \mid kX = -X\}$$

for any tangent vector X of M. D_1 and D_2 are parallel and involutive because of (2.22). Thus, there exist maximal integral manifolds for D_1 and D_2 respectively, which are totally geodesic in M.

In usual way, M is a product of two spheres $S^{n-1} \times S^n$ provided that M is complete. Therefore, Theorem 2.3 is proved.

Replacing the assumption (2.15) in Theorem 2.3 by

$$l_c^e f_e^a - f_c^e l_e^a = 0,$$

we can easily see that the hypersurface is totally geodesic.

Thus we have

COROLLARY 2.4. *Let M be a k-invarinat hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2}) (n > 1)$ satisfying*

$$l_c^e f_e^a - f_c^e l_e^a = 0.$$

Then M is totally geodesic. Moreover, the hypersurface is complete and M is $S^{n-1} \times S^n$.

THEOREM 2.5. *Let M be a compact orientable k-invarinat hypersurface of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})(n>1)$ with constant mean curvature. If the function μ has definite sign on M,then M is totally geodesic and consequently $S^{n-1}\times S^n$.*

Proof. Since the hypersurface M is k-invariant, (1.8) and (1.9) imply respectively that

$$\nabla_e u^e=\mu l+\lambda\alpha,\quad \nabla_e w^e=-2n\mu$$

because of (1.3).

Therefore, applying the Green-Stokes theorem to the above equation, we obtain

$$\int\lambda\alpha\mathrm{d}\sigma=0$$

because the mean curvature of M is constant, $\mathrm{d}\sigma$ being the volume element of M.

We have from (1.12)

$$k_{be}l_c^e=\alpha l_{cb} \tag{2.24}$$

provided that hypersurface is k-invariant.

Operating ∇^c to the second equation of (1.10) with $k_c=0$, we find

$$\Delta\mu-\nabla_e w^e=-(\nabla^c l_{ce})u^e-l^{cb}(\nabla_c u_b), \tag{2.25}$$

where Δ means the Laplacian operator.

On the other hand, the function l being constant, (1.7) with $k_c=0$ yields

$$\nabla^c l_{ce}=0.$$

Using this fact, (2.25) leads to

$$\Delta\mu-\nabla_e w^e=-\mu l_{cb}l^{cb}+\lambda\alpha l,$$

with the aid of (1.8) and (2.24).

Integrating this on M and making use of (2.23), we get

$$\int\mu l_{cb}l^{cb}\mathrm{d}\sigma=0$$

because the mean curvature of M is constant.

Since the function μ has definite sign, we have

$$\mu l_{cb} l^{cb} = 0. \tag{2.26}$$

If we transvect (2.24) with w^b and use (2.3), we find

$$l_{ce} w^e = 0 \tag{2.27}$$

since (2.3) is a direct consequence of (1.5) with $k^a = 0$.

If the function $l_{cb}l^{cb}$ does not vanish at some point p in M, then gives $\mu(p) = 0$ and hence $\nu(p) = 0$ because of (2.4) with $k_c = 0$.

Therefore the second equation of (1.10) turned out to be

$$l_{ce} u^e = wc \tag{2.28}$$

at the point p. So (2.27) and (2.28) mean that $w_c = 0$ at p of M. It contradicts the fact that

$$w_e w^e = 1 - \mu^2 - \nu^2$$

at the point p. Thus, it follows that the hypersurface is totally geodesic because of (2.26).

According to Theorem 2.3, our assertion is true.

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