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WEAK CONVERGENCE THEOREMS OF ASYMPTOTICALLY NONEXPANSIVE SEMIGROUPS IN BANACH SPACES*

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1. Introduction

In [16], Opial obtained the weak convergence theorem in a Hilbert space; Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive asymptotically regular mapping for which the set $\mathcal{F}(T)$ of fixed points is nonempty. Then, for any x in C, the sequence $\{T^n x\}$ is weakly convergent to an element of $\mathcal{F}(T)$ (cf. [2],[17]). Similar results were also obtained by Bruck ([3]), Emmanuele ([4]), Gornicki ([6]), Hirano ([7]), Kobayashi ([9]) and Miyadera ([14]) in uniformly convex Banach spaces. Corresponding theorems for asymptotically nonexpansive mappings and asymptotically nonexpansive semigroups were investigated by many mathematicians ([1], [13], [15], [19], [20], [21]).

Recently, Lin-Tan- Xu ([11]) proved the convergence of iterates $\{T^n x\}$ of an asymptotically nonexpansive mapping T in Banach spaces without the uniform convexity.

And also, Lau-Takahashi ([10]) proved the following theorem; Let C be a closed convex subset of a uniformly convex Banach space X with Fréchet differentiable norm, G a right reversible semitopological semigroup, and $\mathcal{S} = \{S(t) : t \in G\}$ a nonexpansive semigroup on C. If $\mathcal{F}(\mathcal{S}) \neq \varnothing$ and $\mathcal{W}(x) \subseteq \mathcal{F}(\mathcal{S})$ for $x \in C$, then the net $\{S(t)x\}$ converges weakly to some $p \in \mathcal{F}(\mathcal{S})$ (see Theorem 2 and 3 in [10]).

In this paper, we prove the result of Lau-Takahashi ([10]) in Banach spaces without the uniform convexity. The results of this paper are also complete extensions of Lin-Tan-Xu ([11]).

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2. Preliminaries and Notations

Let C be a nonempty closed convex subset of a real Banach space X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* ([5]) if for each $n \geq 1$,

$$\| T^n x - T^n y \| \leq (1 + \alpha_n) \| x - y \|$$

for all $x, y \in C$, where $\lim_{n\to\infty} \alpha_n = 0$. In particular if $\alpha_n = 0$ for all $n \geq 1$, then T is said to be *nonexpansive*. Let $\mathcal{S} = \{S(t) : t \geq 0\}$ be a family of mappings from C into itself. $\mathcal{S}$ is called an *asymptotically nonexpansive semigroup* on C if $S(t+s) = S(t)S(s)$ for every $t, s \geq 0$, and there exists a function $\alpha(\cdot) : R^+ \to R^+$ with $\lim_{t\to\infty} \alpha(t) = 0$ such that

$$\| S(t)x - S(t)y \| \leq (1 + \alpha(t)) \| x - y \|$$

for all $x, y \in C$ and $t \geq 0$. In particular, if $\alpha(t) = 0$ for all $t \geq 0$, then $\mathcal{S}$ is called a *nonexpansive semigroup* on C.

Let G be a semitopological semigroup, i.e., G is a semigroup with a Hausdorff topology such that for each $s \in G$ the mappings $s \to as$ and $s \to sa$ from G to G are continous. G is called *right reversible* if any two closed left ideals of G have nonvoid intersection. In this case, $(G, \succcurlyeq)$ is a directed system when the binary relation " $\succcurlyeq$ " on G is defined by $t \succcurlyeq s$ if and only if

$$\{t\} \cup \overline{Gt} \subseteq \{s\} \cup \overline{Gs}$$

for all $t, s \in G$. Right reversible semitopological semigroup include all commutative semigroups which are right amenable as discrete semigroups ([8]). Left reversibility of G is defined similarly. G is called *reversible* if it is both left and right reversible.

A family $\mathcal{S} = \{S(t) : t \in G\}$ of mappings from C into itself is said to be a *continuous representation of* G on C if it satisfies the followings:

(1) $S(ts)x = S(t)S(s)x$ for all $t, s \in G$ and $x \in C$.

(2) For every $x \in C$, the mapping $(s, x) \to S(s)x$ from $G \times C$ into C is continuous when $G \times C$ has the product topology.

A continuous representation $\mathcal{S}$ of G on C is said to be an *asymptotically nonexpansive semigroup* on C if each $t \in G$, there exists $k_t > 0$ such that

$$\| S(t)x - S(t)y \| \leq (1 + k_t) \| x - y \|$$

for all $x, y \in C$, where $\lim_{t\in G} k_t = 0$. Let $\mathcal{F}(\mathcal{S})$ denote the set of all common fixed points of mappings $S(t)$ for $t \in G$ in C, that is,

$$\mathcal{F}(\mathcal{S}) = \bigcap_{t\in G} \mathcal{F}(S(t)).$$

Some rudiments in the geometry of Banach spaces are necessary for the proofs of the main theorems in this paper. In the sequel, we give the notations; $\overline{\lim} = \limsup$, $\underline{\lim} = \liminf$, " $\rightharpoonup$ " for weak convergence, and " $\to$ " for strong convergence. Also, a space X is always understood to be an infinite dimensional Banach space without Schur's property, i.e., the weak and strong convergence doesn't coincide for nets.

A Banach space X is said to satisfy *Opial's condition* if for each net $\{x_\alpha\}_{\alpha\in G}$ in X, the condition $x_\alpha \rightharpoonup x$ implies that

$$\overline{\lim_{\alpha\in G}} \| x_\alpha - x \| < \overline{\lim_{\alpha\in G}} \| x_\alpha - y \|$$

for all $y \neq x$ (see [16] for any sequence $\{x_n\}$ in X). Spaces possessing that property include the Hilbert spaces and the l^p spaces for $1 \leq p < \infty$. However, $L^p (p \neq 2)$ do not satisfy that property ([12]).

Recently, Prus ([18]) introduced the notion of the uniform Opial condition for any sequence $\{x_n\}$ in X. A Banach space X is said to satisfy the *uniform Opial condition* if for each $c > 0$, there exists an $r > 0$ such that

$$1 + r \leq \underline{\lim}_{\alpha\in G} \| x + x_\alpha \|$$

for each $x \in X$ with $\| x \| \geq c$ and each net $\{x_\alpha\}_{\alpha\in G}$ in X such that $x_\alpha \rightharpoonup 0$ as $\alpha \in G$ and $\underline{\lim}_{\alpha\in G} \| x_\alpha \| \geq 1$. We now define *Opial's modulus of* X, denoted by $r_X(\cdot)$, as follows

$$r_X(c) = \inf\{\underline{\lim}_{\alpha\in G} \| x + x_\alpha \| - 1\},$$

where $c \geq 0$ and the infimum is taken over all $x \in X$ with $\| x \| \geq c$ and nets $\{x_\alpha\}_{\alpha\in G}$ in X such that $\lim_{\alpha\in G} x_\alpha = 0$ weakly and $\underline{\lim}_{\alpha\in G} \| x_\alpha \| \geq 1$. It is easy to see that the function $r_X(\cdot)$ is nondecreasing and that X satisfies the uniform Opial condition if and only if $r_X(c) > 0$ for all $c > 0$. Furthermore, we know that the Opial's modulus $r_X(\cdot)$ of X is continuous.

We now introduce the notion of the locally uniform Opial condition. A Banach space X is said to satisfy the *locally uniform Opial condition* if for any weak null net $\{x_\alpha\}_{\alpha\in G}$ in X with $\underline{\lim}_{\alpha\in G} \| x_\alpha \| \geq 1$ and any $c > 0$, there is an $r > 0$ such that

$$1 + r \leq \underline{\lim}_{\alpha\in G} \| x_\alpha + x \|$$

for all $x \in X$ with $\| x \| \geq c$ (see [11] for sequence). We can easily see that each "$\underline{\lim}$" can be replaced by "$\overline{\lim}$" in the definition of the locally uniform Opial condition. Clearly, uniform Opial condition implies locally uniform Opial condition, which in turn implies Opial's condition ([11]).

Let D be a subset of a Banach space X, then $\overline{conv}D$ will denote its closed convex hull of D. Let $\mathcal{W}(x)$ denote the set of all weak limits of subnets $\{S(t_\alpha)x\}_{\alpha\in G}$ of the net $\{S(t)x\}$ for a semitopological semigroup G.

3. Locally Uniform Opial and Uniform Opial Condition

In this section, we study the asymptotic behavior of the orbits $\{S(t)x\}$ for an asymptotically nonexpansive semigroup $\mathcal{S} = \{S(t) : t \in G\}$, under the locally uniform Opial condition or uniform Opial condition.

We have the following equivalent statements for locally uniform Opial condition.

PROPOSITION 3.1. *If X is a Banach space, then the following two statements are equivalent.*

(1) *X satisfies the locally uniform Opial condition.*
(2) *For any net $\{x_\alpha\}_{\alpha\in G}$ in X which converges weakly to $x \in X$ as $\alpha \in G$ and for any net $\{y_\beta\}_{\beta\in G}$ in X, if*

$$\overline{\lim_{\beta\in G}}(\overline{\lim_{\alpha\in G}} \| x_\alpha - y_\beta \|) \leq \overline{\lim_{\alpha\in G}} \| x_\alpha - x \|,$$

then $\{y_\beta\}$ converges strongly to $x \in X$ as $\beta \in G$.

Proof. (1)$\Rightarrow$(2). Assume that X satisfies the locally uniform Opial condition. Let $\{x_\alpha\} \subset X$ be a weakly convergent net to x and $\{y_\beta\}$ a net in X such that

$$\overline{\lim_{\beta\in G}}(\overline{\lim_{\alpha\in G}} \| x_\alpha - y_\beta \|) \leq \overline{\lim_{\alpha\in G}} \| x_\alpha - x \| .$$

If $\overline{\lim}_{\alpha\in G} \| x_\alpha - x \|= 0$, then we have $\lim_{\alpha\in G} \| x_\alpha - x \|= 0$. So it is obvious that $\{y_\beta\}$ converges strongly to $x \in X$ as $\beta \in G$. And if $\overline{\lim}_{\alpha\in G} \| x_\alpha - x \|> 0$, then there exists a subnet $\{x_{\alpha_\gamma}\}$ of $\{x_\alpha\}$ such that

$$\lim_{\gamma\in G} \| x_{\alpha_\gamma} - x \|= \overline{\lim_{\alpha\in G}} \| x_\alpha - x \| \ \ (\equiv b).$$

If $\{y_\beta\}$ dose not converge strongly to x, then there exist an $\varepsilon_o > 0$ and a subnet $\{y_{\beta_\delta}\}$ of $\{y_\beta\}$ such that

$$\| x - y_{\beta_\delta} \|\geq \varepsilon_o$$

for all $\delta \in G$. Letting $z_\gamma = \dfrac{x_{\alpha_\gamma} - x}{b}$. Then we have $\lim_{\gamma\in G} \| z_\gamma \|= 1$. Hence, for all $z \in X$ with $\| bz \|\geq \varepsilon_o$, there exists an $r > 0$ such that

$$1 + r \leqslant \underline{\lim_{\gamma\in G}} \| z_\gamma + z \|$$

from the definition of the locally uniform Opial condition. So, we have

$$1 + r \leq \underline{\lim_{\gamma\in G}} \left\| \frac{x_{\alpha_\gamma} - x}{b} + \frac{x - y_{\beta_\delta}}{b} \right\|,$$

that is,

$$b(1 + r) \leqslant \underline{\lim_{\gamma\in G}} \| x_{\alpha_\gamma} - y_{\beta_\delta} \|$$

for all $\delta \in G$, which implies

$$\overline{\lim_{\beta\in G}}\left(\overline{\lim_{\alpha\in G}} \| x_\alpha - y_\beta \| \right)\geq b(1 + r) > \overline{\lim_{\alpha\in G}} \| x_\alpha - x \| .$$

This is a contradiction.

(2)⇒(1). Suppose that X does not satisfy the locally uniform Opial condition. Then there exist a weakly null net $\{x_\alpha\}_{\alpha\in G}$ in X with $\overline{\lim}_{\alpha\in G} \| x_\alpha \| \geqslant 1$, a constant $c > 0$ and a net $\{y_\beta\} \subset X$ with $\| y_\beta \| \geqslant c$ for all $\beta \in G$ such that

$$1 + r_\beta > \overline{\lim_{\alpha\in G}} \| x_\alpha + (-y_\beta) \|$$

for every $r_\beta > 0$ with $\lim_{\beta\in G} r_\beta = 0$. Hence we have

$$\overline{\lim_{\beta\in G}} \left(\overline{\lim_{\alpha\in G}} \| x_\alpha - y_\beta \| \right) \leq 1 \leq \overline{\lim_{\alpha\in G}} \| x_\alpha \| .$$

By assumption, $\{y_\beta\}$ converges strongly to 0. This contracts the fact $\| y_\beta \| \geq c$ for all $\beta \in G$. This completes the proof.

We begin with the following result which is crucial for the Proposion 3.3.

LEMMA 3.2. *Let C be a nonempty weakly compact convex subset of X satisfying the Opial's condition and let G be a right reversible semitopological semigroup. Let $S = \{S(t) : t \in G\}$ be an asymptotically nonexpansive semigroup on C. If also $\{x_\alpha\}_{\alpha\in G}$ is a net in C converging weakly to x and for which the net $\{x_\alpha - S(t)x_\alpha\}$ converges strongly to 0 as $\alpha \in G$ for all $t \in G$. Then we have the following results.*

(1) $\bigcap_{t\in G} \overline{conv}\{S(s)x : s \succcurlyeq t\} = \overline{conv}\,\mathcal{W}(x) \neq \varnothing$.
(2) $\bigcap_{t\in G} \overline{conv}\{S(s)x : s \succcurlyeq t\} = \{x\}$.

Proof. (1). Since C is weakly compact, $\bigcap_{t\in G} \overline{conv}\{S(s)x : s \succcurlyeq t\} \neq \varnothing$ and

$$\bigcap_{t\in G} \overline{conv}\{S(s)x : s \succcurlyeq t\} = \overline{conv}\,\mathcal{W}(x).$$

In fact, put $M_t = \overline{conv}\{S(s)x : s \succcurlyeq t\}$. Then the inclusion $\mathcal{W}(x) \subseteq \cap_{t\in G} M_t (\equiv M)$ being trivial, and hence $\overline{conv}\,\mathcal{W}(x) \subseteq M$. Now we must prove that $M \subseteq \overline{conv}\,\mathcal{W}(x)$. If not, then there is an $u \in M - \overline{conv}\,\mathcal{W}(x)$. So, there exists a bounded linear functional B such that

$$B(u) > \sup\{B(v) : v \in \overline{conv}\,\mathcal{W}(x)\} \geq \sup\{B(v) : v \in \mathcal{W}(x)\}.$$

Furthermore, since $u \in M_t$ for all $t \in G$,

$$B(u) \leq \sup\{B(v) : v \in M_t, t \in G\} \leq \sup\{B(S(s)x) : s \succcurlyeq t\}.$$

Hence, we have $B(u) \leq \overline{\lim}_{t \in G} B(S(t)x)$. Taking a subnet of $\{B(S(t)x)\}$ which converges to the $\overline{\lim}_{t \in G} B(S(t)x)$ and using the Eberliein-Smulyan theorem, there exists a subnet $\{S(t_\alpha)x\}$ of $\{S(t)x\}$ such that $S(t_\alpha)x \rightharpoonup w$ and $B(u) \leq B(w)$. Since $w \in \mathcal{W}(x)$, we have a contradiction.

(2). Suppose that there exists $y \in \bigcap_{t \in G} \overline{conv}\{S(s)x : s \succcurlyeq t\}$ such that $y \neq x$. Then by Opial's condition,

$$\overline{\lim_{\alpha \in G}} \| x_\alpha - x \| < \overline{\lim_{\alpha \in G}} \| x_\alpha - y \| .$$

Since $\lim_{t \in G} k_t = 0$, there is a $t_\circ \in G$ such that for all $t \in G$ with $t \succcurlyeq t_\circ$,

$$k_t < \frac{\overline{\lim}_{\alpha \in G} \| x_\alpha - y \| - \overline{\lim}_{\alpha \in G} \| x_\alpha - x \|}{\overline{\lim}_{\alpha \in G} \| x_\alpha - x \| + 1}.$$

Since $y \in \overline{conv}\{S(s)x : s \succcurlyeq t\}$ for all $t \succcurlyeq t_\circ$, there exist $\lambda_i (i = 1, 2, \cdots, n) \geqslant 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that

$$\| \textstyle\sum_{i=1}^n \lambda_i S(s_i)x - y \| < \frac{\overline{\lim}_{\alpha \in G} \| x_\alpha - y \| - \overline{\lim}_{\alpha \in G} \| x_\alpha - x \|}{\overline{\lim}_{\alpha \in G} \| x_\alpha - x \| + 1}$$

for all $s_i \succcurlyeq t (i = 1, 2, \cdots, n)$. Hence, we have

$$\begin{aligned}
\overline{\lim_{\alpha \in G}} \| x_\alpha - y \| &\leq \overline{\lim_{\alpha \in G}} \| x_\alpha - \sum_{i=1}^n \lambda_i S(s_i)x \| + \| \sum_{i=1}^n \lambda_i S(s_i)x - y \| \\
&\leq \| \sum_{i=1}^n \lambda_i S(s_i)x - y \| + \sum_{i=1}^n \lambda_i \overline{\lim_{\alpha \in G}} \| S(s_i)x_\alpha - S(s_i)x \| \\
&\leq \| \sum_{i=1}^n \lambda_i S(s_i)x - y \| + \sum_{i=1}^n \lambda_i (1 + k_{s_i}) \overline{\lim_{\alpha \in G}} \| x_\alpha - x \| \\
&< \frac{(\overline{\lim}_{\alpha \in G} \| x_\alpha - y \| - \overline{\lim}_{\alpha \in G} \| x_\alpha - x \|)}{\overline{\lim}_{\alpha \in G} \| x_\alpha - x \| + 1} \\
&\quad + \left(\frac{\overline{\lim}_{\alpha \in G} \| x_\alpha - y \| + 1}{\overline{\lim}_{\alpha \in G} \| x_\alpha - x \| + 1} \right) \overline{\lim_{\alpha \in G}} \| x_\alpha - x \| \\
&= \overline{\lim_{\alpha \in G}} \| x_\alpha - y \| .
\end{aligned}$$

This contradiction establishes the result (2). This completes the proof.

In Proposition 3.3, we prove the demiclosedness principle at zero for an asymptotically nonexpansive semigroup in a Banach space with the locally uniform Opial codition. The following proposition plays a crucial role in the proofs of our main theorems in this section.

PROPOSITION 3.3. *Let X be a Banach space satisfying the locally uniform Opial condition, C a nonempty weakly compact convex subset of X, and G a right reversible semitopological semigroup. If $S = \{S(t) : t \in G\}$ is an asymptotically nonexpansive semigroup on C. Then $I - S(t)$ is demiclosed at zero, i.e., if $\{x_\alpha\}_{\alpha \in G}$ is a net in C which converges weakly to x as $\alpha \in G$ and if the net $\{x_\alpha - S(t)x_\alpha\}$ converges strongly to zero as $\alpha \in G$, then $(I - S(t))x = 0$ for all $t \in G$.*

Proof. Let $x_\alpha \rightharpoonup x$ and $x_\alpha - S(t)x_\alpha \to 0$ as $\alpha \in G$ for all $t \in G$. By Lemma 3.2, we have $S(t)x \rightharpoonup x$ as $t \in G$. Since $\mathcal{S} = \{S(t) : t \in G\}$ is an asymptotically nonexpansive semigroup on C,

$$\begin{aligned}
\overline{\lim_{s\in G}}(\overline{\lim_{t\in G}} \parallel S(t)x - S(s)x \parallel) &= \overline{\lim_{s\in G}}(\overline{\lim_{t\in G}} \parallel S(st)x - S(s)x \parallel) \\
&\leq \overline{\lim_{s\in G}}(\overline{\lim_{t\in G}}(1 + k_s) \parallel S(t)x - x \parallel) \\
&= \overline{\lim_{t\in G}} \parallel S(t)x - x \parallel .
\end{aligned}$$

So, $\{S(s)x\}$ converges strongly to x as $s \in G$ from Proposition 3.1. Therefore, for all $t \in G, (I - S(t))x = 0$ by the continuity of $S(t)$. This completes the proof.

We need the following lemma in order to prove our main theorems in this section.

LEMMA 3.4. *Let the assumptions in Proposition 3.3 be satisfied. If $S(t)$ is asymptotically regular at some $x \in C$, i.e., $\lim_{t\in G} \parallel S(st)x - S(t)x \parallel = 0$ for all $s \in G$. Then we have the following conclusions.*

(1) $\mathcal{F}(\mathcal{S}) \subseteq E(x)$, *where* $E(x) = \{y \in C : \lim_{t\in G} ||S(t)x - y|| \, exist\}$.

(2) $\mathcal{W}(x) \subseteq \mathcal{F}(\mathcal{S})$.

(3) *$\mathcal{W}(x)$ is a singleton, and hence $\{S(t)x\}$ converges weakly to a point of $\mathcal{F}(\mathcal{S})$.*

Proof. (1). Let $t \succcurlyeq s$ for $t, s \in G$. Then $t \in \{s\} \cup \overline{Gs}$. We may assume that $t \in \overline{Gs}$. So there exists an $g_\alpha \in G$ such that $g_\alpha s \to t$ as $\alpha \in G$. Then, for $\alpha \in G$ and $y \in \mathcal{F}(\mathcal{S})$,

$$\begin{aligned} \| S(g_\alpha s)x - y \| &= \| S(g_\alpha)S(s)x - S(g_\alpha)y \| \\ &\leq (1 + k_{g_\alpha}) \| S(s)x - y \| . \end{aligned}$$

Hence, we have

$$\| S(t)x - y \| \leq \| S(s)x - y \|$$

for all $t \succcurlyeq s$ and $y \in \mathcal{F}(\mathcal{S})$. This proves that $\mathcal{F}(\mathcal{S}) \subseteq E(x)$ as desired.

(2). Let $\{S(t_\alpha)x\}$ be a subnet of $\{S(t)x\}$ converging weakly to $y \in C$ as $\alpha \in G$. Letting $x_\alpha = S(t_\alpha)x$. Then, since $S(s)$ is asymptotically regular, $\| x_\alpha - S(s)x_\alpha \| \to 0$ as $\alpha \in G$ for all $s \in G$. Since $I - S(s)$ is demiclosed at zero, from Proposition 3.3, $(I - S(s))y = 0$ for all $s \in G$. This completes the proof of (2).

(3). Let y_1 and y_2 be two weak limits of subnets $\{S(t_\alpha)x\}$ and $\{S(t_\beta)x\}$ of the net $\{S(t)x\}$, respectively. Since $\mathcal{W}(x) \subseteq \mathcal{F}(\mathcal{S})$, there are $d_1, d_2 \geq 0$ by (1) such that

$$d_1 = \lim_{t \in G} \| S(t)x - y_1 \| \quad \text{and} \quad d_2 = \lim_{t \in G} \| S(t)x - y_2 \| .$$

If $y_1 \neq y_2$, then we have

$$\begin{aligned} d_1 &= \lim_{t \in G} \| S(t)x - y_1 \| = \overline{\lim_{\alpha \in G}} \| S(t_\alpha)x - y_1 \| \\ &< \overline{\lim_{\alpha \in G}} \| S(t_\alpha)x - y_2 \| = \overline{\lim_{\beta \in G}} \| S(t_\beta)x - y_2 \| \\ &< \overline{\lim_{\beta \in G}} \| S(t_\beta)x - y_1 \| = \lim_{t \in G} \| S(t)x - y_1 \| \\ &= d_1. \end{aligned}$$

This is a contradiction, which implies that $\mathcal{W}(x)$ is a singleton. This completes the proof.

As a direct consequence of Lemma 3.4, we can prove the convergence theorems of orbits $\{S(t)x\}$ of an asymptotically nonexpansive semigroup $\mathcal{S} = \{S(t) : t \in G\}$ for a right reversible semitopological semigroup G.

THEOREM 3.5. *Let C be a nonempty weakly compact convex subset of a Banach space X satisfying the locally uniform Opial condition, G a right reversible semitopological semigroup, and $\mathcal{S} = \{S(t) : t \in G\}$ an asymptotically nonexpansive semigroup on C. If $S(t)$ is asymptotically regular at $x \in C$, then $\{S(t)x\}$ converges weakly to a point p in $\mathcal{F}(\mathcal{S})$ as $t \in G$.*

Proof. From (2),(3) of Lemma 3.4, it is easy to show that the orbits $\{S(t)x\}$ converges weakly to p in $\mathcal{F}(\mathcal{S})$ as $t \in G$.

It is not clear whether the asymptotic regularity in Theorem 3.5 can be weakened to the weakly asymptotic regularity. We improve the Theorem 3.5 when the space X is assumed to be satisfying the uniform Opial condition.

THEOREM 3.6. *Let C be a nonempty weakly compact convex subset of a Banach space X satisfying the uniform Opial condition and let $G, \mathcal{S}$ be as in Theorem 3.5. If $S(t)$ is weakly asymptotically regular at $x \in C$, i.e., $w\text{-}\lim_{t\in G} \| S(st)x - S(t)x \| = 0$ for all $s \in G$, then $\{S(t)x\}$ converges weakly to a point p in $\mathcal{F}(\mathcal{S})$.*

Proof. In order to prove the Theorem 3.6, we must prove the results of Lemma 3.4 under assumptions of Theorem 3.6. We can easily prove that (1) and (3) of Lemma 3.4 are obvious. Now, we have to show only (2). Let y be a weak limit of subnet $\{S(t_\alpha)x\}$ of $\{S(t)x\}$ as $\alpha \in G$. Since $S(t)$ is weakly asymptotically regular at $x \in C$, $\{S(st_\alpha)x\}$ weakly converges to y as $\alpha \in G$. Letting

$$r(s) = \overline{\lim_{\alpha \in G}} \| S(st_\alpha)x - y \| .$$

Then, we have

$$\lim_{s\in G} r(s) = \inf_{s\in G} r(s) \ (\equiv r).$$

In fact, for given $\varepsilon > 0$, there exists an $s_0 \in G$ such that $r(s_0) < r + \frac{\varepsilon}{2}$. Also, since $\lim_{t\in G} k_t = 0$, there exists a $t_0 \in G$ such that $k_t < \frac{\varepsilon}{2M}$ for all $t \succcurlyeq t_0$, where $M = \sup_{s\in G} \|S(s)x - y\|$. Let $\beta \succcurlyeq \alpha_0 = t_0 s_0$. Since G is right reversible, $\beta \in \{\alpha_0\} \cup \overline{G\alpha_0}$ and so, we may assume that $\beta \in \overline{G\alpha_0}$. Hence, there exists $\{t_\gamma\} \subset G$ such that $t_\gamma \alpha_0 \to \beta$ as $\gamma \in G$.

Since $S(t_\gamma \alpha_0 t_\alpha)x \rightharpoonup y$ for all $\gamma \in G$, from the Opial's condition,

$$\begin{aligned} r(t_\gamma \alpha_0) &= \overline{\lim_{\alpha \in G}} \parallel S(t_\gamma \alpha_0 t_\alpha)x - y \parallel \\ &\leq \overline{\lim_{\alpha \in G}} \parallel S(t_\gamma t_0 s_0 t_\alpha)x - S(t_\alpha t_0)y \parallel \\ &\leq (1 + k_{t_\gamma t_0}) r(s_0) \\ &\leq r(s_0) + \frac{\varepsilon}{2} \end{aligned}$$

for all $\gamma \in G$. Hence, we have

$$r(\beta) \leq r(s_0) + \frac{\varepsilon}{2}$$

for all $\beta \succcurlyeq \alpha_0$. Therefore, we have

$$\overline{\lim_{s \in G}} r(s) \leq \sup_{\beta \succcurlyeq \alpha_0} r(\beta) < r + \varepsilon = \inf_{s \in G} r(s) + \varepsilon.$$

Since ε is arbitrary, we have

$$r = \lim_{s \in G} r(s) \ \text{ and } \ r \leq r(s)$$

for all $s \in G$.

First, if $r = 0$, then $\lim_{s \in G} S(s)y = y$ from

$$\begin{aligned} \|S(s)y - y\| &\leq \overline{\lim_{\alpha \in G}} \{\parallel S(s)y - S(st_\alpha)x \parallel + \parallel S(st_\alpha)x - y \parallel\} \\ &\leq (1 + k_s) \overline{\lim_{\alpha \in G}} \parallel y - S(t_\alpha)x \parallel + r(s). \end{aligned}$$

Therefore, we have $S(t)y = y$ for all $t \in G$.

Now suppose that $r > 0$. In order to get the desired result, it suffices to show that $\{S(t)y\}$ converges strongly to y. If not, there exist an $\varepsilon > 0$ and $\beta \in G$ such that $\|S(t_\beta)y - y\| \geq \varepsilon$. Since $\lim_{s \in G} r(s) = r(\equiv \inf r(s))$, there exist an $s_0 \in G$ such that $r(s_0) < r(1 + r_X(c))$, where $r_X(c)$ is the Opial's modulus of X and $c = \frac{\varepsilon}{r} (> 0)$. And also, we know that, for each $\beta \in G$,

$$\frac{S(t_\beta s_0 t_\alpha)x - y}{r} \rightharpoonup 0$$

as $\alpha \in G$ and

$$\overline{\lim_{\alpha\in G}}\left\|\frac{S(t_\beta s_0 t_\alpha)x-y}{r}\right\| = \frac{r(t_\beta s_0)}{r} \geq 1$$

with $\frac{\|y-S(t_\beta)y\|}{r} \geq c$. Since $r_X(c) > 0$, we have

$$1 + r_X(c) \leq \overline{\lim_{\alpha\in G}}\left\|\frac{S(t_\beta s_0 t_\alpha)x-y}{r} + \frac{y-S(t_\beta)y}{r}\right\|.$$

On ther other hand, from

$$\overline{\lim_{\alpha\in G}}\,\|S(t_\beta s_0 t_\alpha)x - S(t_\beta)y\| \leq (1+k_{t_\beta})\,\overline{\lim_{\alpha\in G}}\,\|S(s_0 t_\alpha)x - y\|,$$

we have

$$1 + r_X(c) > \frac{r(s_0)}{r} \geq \overline{\lim_{\beta\in G}}\,\overline{\lim_{\alpha\in G}}\left\|\frac{S(t_\beta s_0 t_\alpha)x-y}{r} + \frac{y-S(t_\beta)y}{r}\right\|.$$

This is a contradiction, which implies that $\{S(t)y\}$ converges strongly to y. Thus we have $\mathcal{W}(x) \subseteq \mathcal{F}(S)$. This completes the proof.

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