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LOCAL EXTREME POINTS AND ISOLATED POINTS OF THE SET OF SCHWARZIANS OF UNIVALENT FUNCTIONS

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1. Introduction

We assume throughout the paper that the function f is holomorphic locally univalent from the open unit disk $\mathbf{D} = \{z : |z| < 1\}$ into the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$; here $\mathbf{C}$ is the complex plane.

The Schwarzian derivative of a function f is defined by

$$S_f(z) = (f''(z)/f'(z))' - (1/2)(f''(z)/f'(z))^2$$

and satisfies the basic transformation law

$$S_{f\circ g}(z) = S_f(g(z))g'(z)^2 + S_g(z)$$

for any conformal map g. And $S_f(z)$ is holomorphic in $\mathbf{D}$. The Schwarzian derivative vanishes identically if and only if f is a Möbius transformation [7].

A region Ω in the $\hat{\mathbf{C}}$ is called hyperbolic if $\hat{\mathbf{C}} - \Omega$ contains at least three points. The hyperbolic metric on $\mathbf{D}$ is

$$\lambda_{\mathbf{D}}(z)|dz| = (1 - |z|^2)^{-1}|dz|.$$

The densiy $\lambda_\Omega(w)$ of the hyperbolic metric $\lambda_\Omega(w)|dw|$ on a hyperbolic region Ω is defined by

$$\lambda_\Omega(f(z))|f'(z)| = (1 - |z|^2)^{-1},$$

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where $w = f(z)$ is any holomorphic universal covering projection of $\mathbf{D}$ onto Ω. The density has the property

$$\lambda_{\mathbf{D}}(T(z))|T'(z)| = \lambda_{\mathbf{D}}(z)$$

for any $T \in \mathrm{Aut}(\mathbf{D})$, the group of conformal automorphisms of $\mathbf{D}$. It follows that

$$|S_{f\circ T}(z)|\lambda_{\mathbf{D}}(z)^{-2} = |S_f(T(z))|\lambda_{\mathbf{D}}(T(z))^{-2}.$$

Thus we can introduce a norm $\|\ \|_{\mathbf{D}}$, where

$$\|\varphi\|_{\mathbf{D}} = \sup_{z\in\mathbf{D}} |\varphi(z)|\lambda_{\mathbf{D}}(z)^{-2},$$

and where $\varphi(z)$ is any holomorphic function in $\mathbf{D}$. Then we have $\|S_{f\circ T}\|_{\mathbf{D}} = \|S_f\|_{\mathbf{D}}$. For a general discussion of the hyperbolic metric we refer the reader to [1] and [6]. We shall need the following basic properties, which are stated without proof.

Assume Ω and Δ are hyperbolic plane regions.

CONFORMAL INVARIANCE. If f is a conformal mapping of Ω onto Δ, then

$$\lambda_{\Delta}(f(z))|f'(z)| = \lambda_{\Omega}(z).$$

PRINCIPLE OF HYPERBOLIC METRIC. If f is holomorphic on Ω and $f(\Omega) \subseteq \Delta$, then

$$\lambda_{\Delta}(f(z))|f'(z)| \leq \lambda_{\Omega}(z).$$

Equality occurs at some point if and only if f is a holomorphic covering of Ω onto Δ.

MONOTONICITY. If $\Omega \subseteq \Delta$, then for $z \in \Omega$, $\lambda_{\Delta}(z) \leq \lambda_{\Omega}(z)$. If equality holds at a single point, then $\Omega = \Delta$.

We define complex Banach space E by

$$E = \{\varphi | \varphi : \mathbf{D} \to \mathbf{C} \text{ holomorphic, } \|\varphi\|_{\mathbf{D}} < \infty\}.$$

Let

$$U = \{S_f : f : \mathbf{D} \to \hat{\mathbf{C}} \ \text{ conformal univalent}\}.$$

In this paper, we investigate certain aspects of the linear and topological structure of U, namely its extreme points and isolated points.

An extreme point of the set U is a function $\varphi \in U$ such that if φ has a convex decomposition $\varphi = t\varphi_1 + (1-t)\varphi_2$ with $0 < t < 1$ and $\varphi_1, \varphi_2 \in U$, then $\varphi_1 = \varphi_2$.

The set of extreme points of U will be denoted by $ext(U)$, and the same notation will be used for the extreme points of other sets. It is important to note that $ext(U)$ is independent of choice of topology; the concept of extreme point is purely algebraic.

An isolated point of U in E is a $\varphi \in U$ for which there exists an $\varepsilon > 0$ such that if $||\psi - \varphi||_{\mathbf{D}} \leq \varepsilon$ with $\psi \in U$, then $\psi = \varphi$.

Clearly, the concept of an isolated point is tied up very intimately with the choice of topology; U does not have any isolated points in $H(\mathbf{D})$, the space of holomorphic functions on $\mathbf{D}$ endowed with the topology of locally uniform convergence.

Now we introduce the concept that subsums both extreme points and isolated points.

A local extreme point of U in E is a $\varphi \in U$ such that there exists a $\varepsilon > 0$ such that

$$\varphi \in ext(U \cap \{\psi \in E : ||\psi - \varphi||_{\mathbf{D}} \leq \varepsilon\}).$$

The set U has been of some interest due to its connection with Bers model

$$Q = \{S_f \in U : f \text{ has qc extention to } \hat{\mathbf{C}}\}$$

of the universal Teichmüller space. It was shown by Ahlfors [8] that Q is open, and the relationship between U and Q was clarified by Gehring [9], who showed that $Q = \mathrm{int}(U)$. It was for a long-standing question of Bers, answered in the negative by Gehring [10], whether $U = \overline{Q}$. The concept of conformal rigid regions sheds more light on the relationship between U and $\overline{Q}$. Because of the change-of-variable formula

$$||S_f - S_g||_{\mathbf{D}} = ||S_{f \circ g^{-1}}||_{g(\mathbf{D})}$$

we see that if g is a conformal map from $\mathbf{D}$ into $\hat{\mathbf{C}}$, then S_g is an isolated point of U in E if and only if $g(\mathbf{D})$ is rigid [13]. In [13], if U has isolated points, then $U - \overline{Q} = U - \overline{\mathrm{int} U} \neq \phi$. To make the above discussion, it is necessary to prove the existence of conformal rigid regions. This was recently done by Thurston [3] for the first time. Consequently, Thurston proved that U has isolated points. In [11], Overholt discussed the isolated points. Motivated by the study of Overholt, we discuss the isolated points.

In Section 2, we give the properties of local extreme points. In Section 3, we obtain the properties of isolated points.

2. Elementary properties of local extreme points

PROPOSITION 2.1. *Let $f,\ g : \mathbf{D} \to \hat{\mathbf{C}}$ be conformal with $f(\mathbf{D}) \subseteq g(\mathbf{D})$. If S_f is an extreme point of U, then so is S_g. If S_f is an isolated point of U in E, then so is S_g. If S_f is a local extreme point of U in E, then so is S_g.*

Proof. It is enough to carry out the proof for a local extreme point.

By assumption $f = g \circ T$ where $T : \mathbf{D} \to \mathbf{D}$ is a conformal automorphism [14, p. 39]. Suppose $S_g = tS_{g_1} + (1-t)S_{g_2}$ with $S_{g_1}, S_{g_2} \in U$, $S_{g_1} \neq S_{g_2}$, $0 < t < 1$, $\|S_g - S_{g_1}\| \leq \varepsilon$ and $\|S_g - S_{g_2}\|_{\mathbf{D}} \leq \varepsilon$. We have

$$\begin{aligned} S_f = S_{g\circ T} &= S_g(T)T'^2 + S_T \\ &= [tS_{g_1}(T) + (1-t)S_{g_2}(T)]T'^2 + S_T \\ &= t[S_{g_1}(T)T'^2 + S_T] + (1-t)[S_{g_2}(T)T'^2 + S_T] \\ &= tS_{g_1\circ T} + (1-t)S_{g_2\circ T}. \end{aligned}$$

It is clear that $S_{g_1\circ T}$, $S_{g_2\circ T} \in U$, and $S_{g_1\circ T} \neq S_{g_2\circ T}$. Furthermore

$$\begin{aligned} \|S_{g\circ T} - S_{g_1\circ T}\|_{\mathbf{D}} &= \|S_g(T)T'^2 + S_T - S_{g_1}(T)T'^2 - S_T\|_{\mathbf{D}} \\ &= \|(S_g(T) - S_{g_1}(T))T'^2\|_{\mathbf{D}} \\ &= \sup_{z\in\mathbf{D}} |S_g(T(z)) - S_{g_1}(T(z))||T'(z)|^2\lambda_{\mathbf{D}}(z)^{-2} \\ &\leq \sup_{z\in\mathbf{D}} |S_g(T(z)) - S_{g_1}(T(z))|\lambda_{\mathbf{D}}(T(z))^{-2} \\ &\leq \sup_{z\in\mathbf{D}} |S_g(z) - S_{g_1}(z)|\lambda_{\mathbf{D}}(z)^{-2} \\ &= \|S_g - S_{g_1}\|_{\mathbf{D}} \leq \varepsilon \end{aligned}$$

by the Schwarz-Pick lemma [1, p. 3]. By the similar method we have $||S_{g\circ T} - S_{g_2\circ T}|| \leq \varepsilon$. Thus we are finished. □

We may use the proposition 2.1 to obtain a restriction on the omitted set of f if S_f is a local extreme point of U in E.

THEOREM 2.2. *If S_f is a local extreme point of U in E, then f cannot omit an open set.*

Proof. Suppose f omits an open set. Then it will in particular omit some closed disk D_o, say. Let T be a Möbius transformation mapping $\mathbf{D}$ onto $\hat{\mathbf{C}} - D_o$. Clearly $f(\mathbf{D}) \subseteq T(\mathbf{D})$, so proposition 2.1 would imply that S_T is a local extreme point of U in E. But $S_T = 0$, so this is false. We can see this by considering the functions $f_p(z) = [(1+z)/(1-z)]^p$ which are univalent for $0 < p \leq 2$. Since $S_{f_p}(z) = 2(1-p^2)(1-z^2)^{-2}$, we note that the zero function cannot be a local extreme point of U in E. □

3. Elementary properties of isolated points

Since extreme points and isolated points are local extreme points, the conclusion in proposition 2.1 is valid for them. In particular, if S_f is an isolated point of U in E, then f cannot omit an open set [3, p. 191]. This can be seen in other ways too. It seems clear that the property of being an isolated point is much stronger than the property of being a local extreme point. So we might hope to obtain a stronger conclusion than that f cannot omit an open set, if S_f is an isolated point of U in the topology of E.

In order to consider isolated points in greater detail, we must define rigid regions. Let $D \subseteq \hat{\mathbf{C}}$ be a region not necessarily simply connected. We assume that D has $\mathbf{D}$ as its universal covering surface; it is then called a hyperbolic region. It is well known [1,p.150] that the only non-hyperbolic regions are $\hat{\mathbf{C}}, \hat{\mathbf{C}} - \{a\}, \hat{\mathbf{C}} - \{a, b\}$.

We may now define the hyperbolic sup-norm of weight -2 on D by

$$||\varphi||_D = \sup_{z\in D} |\varphi(z)|\lambda_D(z)^{-2},$$

where φ is any holomophic function on D. This norm is analogous in the norm $||\varphi||_D = ||\varphi||_{\mathbf{D}}$.

The following definition is due to Thurston [3].

DEFINITION 3.1. *A hyperbolic region $D \subseteq \hat{\mathbf{C}}$ is rigid if there exists a constant $s = s(D) > 0$ such that if f is a conformal map from $\mathbf{D}$ onto $\hat{\mathbf{C}}$ and $||S_f||_D \leq s$, then f is a Möbius transformation.*

THEOREM 3.2. *Let $D_1, D_2 \subseteq \hat{\mathbf{C}}$ be regions, with $D_1 \subseteq D_2$. If D_1 is rigid then D_2 is rigid.*

Proof. If D_2 is not-hyperbolic, then assertion is true by definition. If D_2 is hyperbolic, then D_1 is hyperbolic. Since $D_1 \subseteq D_2$, we have $\lambda_{D_2}(z) \leq \lambda_{D_1}(z)$ by the monotonicity of the hyperbolic metric. Now if f is a conformal map from D_2 into $\hat{\mathbf{C}}$, then from

$$\begin{aligned} ||S_f||_{D_1} &= \sup_{z \in D_1} |S_f(z)| \lambda_{D_1}(z)^{-2} \leq \sup_{z \in D_1} |S_f(z)| \lambda_{D_2}(z)^{-2} \\ &\leq \sup_{z \in D_2} |S_f(z)| \lambda_{D_2}(z)^{-2} = ||S_f||_{D_2} \end{aligned}$$

and the rigidity of D_1, we conclude that if $||S_f||_{D_2} \leq s(D_1)$, then $f|_{D_1}$ is a Möbius transformation, and so f is a Möbius transformation, and the proof is complete. □

REMARK. A region is said to be simply connected if its complement is connected. Theorem 3.2 is useful when investigating the properties of the complement Γ of a rigid region $D; \Gamma = \hat{\mathbf{C}} - D$. In particular, one sees that the components of Γ can be only points or complements of rigid, simply connected, hyperbolic regions. For if $\Gamma_\circ$ is a component of Γ, then $D_\circ = \hat{\mathbf{C}} - \Gamma_\circ$ is simply connected, and also hyperbolic unless $\Gamma_\circ$ reduces to a point. Now since $D \subseteq D_\circ$, theorem 3.2 shows that $D_\circ$ is rigid.

In [3] Thurston proved the existence of rigid, simply connected, hyperbolic regions by construction. His examples are complements of quasiarcs, and so the complements of these rigid regions automatically have zero area. We know that the last assertion is valid for the complement of any rigid region. In the following, $m(A)$ will always denote the lebesgue planar measure of a set A. One result, however, is following.

THEOREM 3.3. *If $\hat{\mathbf{C}} - \Gamma$ is conformal rigid and Γ is a connected, then $m(\Gamma) = 0$.*

Proof. Suppose $m(\Gamma) > 0$. By a theorem of Ngugen [4], there exist a bounded Lipschitz function ψ on $\hat{\mathbf{C}} - \Gamma$ which is not a constant. Put

$D = \hat{\mathbf{C}} - \Gamma$, and let f be defined on D by $f(z) = z + \varepsilon\psi(z)$. For ε small enought, f is conformal on D. We have

$$S_f = (\varepsilon\psi''' + \varepsilon^2\psi'\psi''' - (3/2)\varepsilon^2\psi''^2)/(1 + \varepsilon\psi')^2.$$

Now by the Lipschitz condition on ψ, we have $\psi'(z) = 0(1)$ in D. So by the Cauchy's estimate [12,p.73] it follows that $\psi''(z) = 0(\delta(z)^{-1})$ and $\psi'''(z) = 0(\delta(z)^{-2})$ in D. Here $\delta(z) = \mathrm{dist}(z, \partial\Gamma)$. For sufficiently small ε, we then get $S_f(z) = 0(\varepsilon\delta(z)^{-2})$. But it is well known that [5,p.45] $(1/4)\delta(z)^{-1} \leq \lambda_D(z) \leq \delta(z)^{-1}$. Thus $|S_f(z)|\lambda_D(z)^{-2} = 0(\varepsilon)$ and so $||S_f||_D = 0(\varepsilon)$, and thus D is not rigid we are finished. $\square$

REMARK. From the above, we see that if S_f is an isolated point of the set of Schwarzians of conformal maps $f : D \to \hat{\mathbf{C}}$, in the topology induced by the norm $||\ ||_D$, then the omitted set of f has zero area. For the details of the connection between rigid regions and isolated Schwarzians, see [3 or 13].

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