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# ON CERTAIN BAZILEVIC FUNCTIONS OF ORDER $\beta$

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## 1. Introduction.

Let $\mathcal{A}(p,n)$ be the class of the functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \qquad (n \in \{1,2,3,\cdots\cdots\}) \tag{1.1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$.

A function $f(z)$ belonging to $\mathcal{A}(p,n)$ is said to be p-valently starlike of order $\beta$ if it satisfies

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \tag{1.2}$$

for some $\beta(0 \leq \beta < p)$ and for all $z \in U$. We denote by $\mathcal{S}^*(p,n,\beta)$ the subclass of $\mathcal{A}(p,n)$ consisting of functions which are p-valently starlike of order $\beta$.

A function $f(z)$ in $\mathcal{A}(p,n)$ is said to be in the subclass $\mathcal{B}(p,n,\alpha,\beta)$ of Bazilevič function class if it satisfies

$$Re\left(\frac{zf'(z)f(z)^{\alpha-1}}{g(z)^{\alpha}}\right) > \beta \tag{1.3}$$

for some $\alpha(0 < \alpha)$ and $\beta(0 \leq \beta < p), g(z) \in \mathcal{S}^*(p,n,0)$ and for all $z \in U$. Further, let $\mathcal{B}_1(p,n,\alpha,\beta)$ be the subclass of $\mathcal{B}(p,n,\alpha,\beta)$ for $g(z) = z^p \in \mathcal{S}^*(p,1,0)$.

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REMARK. *1.* $B_1(1,\alpha,\beta) = \mathcal{B}_1(1,1,\alpha,\beta)$ *were introduced and studied by Owa and Obradovič ([4]) and* $B_1(1,\alpha,0) = \mathcal{B}_1(1,1,\alpha,0)$ *by Singh ([6]).*

*2.* $B(n,\alpha,\beta) = \mathcal{B}_1(1,n,\alpha,\beta)$ *by Ponnusamy ([5]),* $B_1(p,\alpha,\beta) = \mathcal{B}_1(p,1,\alpha,\beta)$ *by Owa([3]) and* $B(1,0) = \mathcal{B}_1(1,n,1,0), B(2,0) = (1,n,2,0)$ *by Cho([1]).*

## 2. The Class $\mathcal{B}_1(p,n,\alpha,\beta)$.

In order to obtain our main result, we recall the following lemmas due to Owa([3]).

LEMMA 1. *If* $f(z) \in B_1(n,\alpha,\beta) = \mathcal{B}_1(1,n,\alpha,\beta)$, *then*

$$Re\left(\frac{f(z)}{z}\right)^{\alpha} > \frac{n+2\alpha\beta}{n+2\alpha} \quad (z \in U). \tag{2. 1}$$

LEMMA 2. *If* $f(z) \in B_1(n,\alpha,\beta) = \mathcal{B}_1(1,n,\alpha,\beta)$, *then*

$$Re\left(\frac{f(z)}{z}\right)^{\frac{\alpha}{2}} > \frac{n+\sqrt{n^2+4\alpha\beta(n+\alpha)}}{2(n+\alpha)} \quad (z \in U). \tag{2. 2}$$

Using the above Lemma 1, we prove the following theorem.

THEOREM 1. *Let* $f(z) \in \mathcal{B}_1(p,n,\alpha,\beta)$ *with* $\alpha > 0$ *and* $0 \leq \beta < p$, *then*

$$Re\left(\frac{f(z)}{z^p}\right)^{\alpha} > \frac{n+2\alpha\beta}{n+2p\alpha} \quad (z \in U). \tag{2. 3}$$

*Proof.* We define the function $h(z)$ by $h(z)^p = f(z)$ for $f(z) \in \mathcal{B}_1(p,n,\alpha,\beta)$. Then we have

$$\frac{zf'(z)f(z)^{\alpha-1}}{z^{p\alpha}} = p\frac{zh'(z)h(z)^{p\alpha-1}}{z^{p\alpha}} \tag{2. 4}$$

Since $f(z) \in \mathcal{B}_1(p,n,\alpha,\beta)$ if and only if $Re\left(\frac{zf'(z)f(z)^{\alpha-1}}{z^{p\alpha}}\right) > \beta$, from (2. 4) we get

$$Re\left(\frac{zh'(z)h(z)^{p\alpha-1}}{z^{p\alpha}}\right) > \frac{\beta}{p}. \tag{2. 5}$$

Thus $h(z) \in \mathcal{B}_1(1, n, p\alpha, \frac{\beta}{p})$. By Lemma 1,

$$Re\left(\frac{f(z)}{z^p}\right)^{\alpha} = Re\left(\frac{h(z)}{z}\right)^{p\alpha}$$

$$> \frac{n + 2\alpha\beta}{n + 2p\alpha}. \tag{2. 6}$$

Letting $n = 1$ in Theorem 1, we have

COROLLARY 1 ([2]). *If $f(z) \in B_(p, \alpha, \beta) = \mathcal{B}_1(p, 1, \alpha, \beta)$ with $\alpha > 0$ and $0 \leq \beta < p$, then*

$$Re\left(\frac{f(z)}{z^p}\right)^{\alpha} > \frac{1 + 2\alpha\beta}{1 + 2p\alpha} \ (z \in U). \tag{2. 7}$$

Letting $p = 1, n = 1$ in Theorem 1, we get

COROLLARY 2([4]). *$f(z) \in B_1(1, \alpha, \beta) = \mathcal{B}_1(1, 1, \alpha, \beta)(\alpha > 0, 0 \leq \beta < p)$ then*

$$Re\left(\frac{f(z)}{z}\right)^{\alpha} > \frac{1 + 2\alpha\beta}{1 + 2\alpha} \ (z \in U). \tag{2. 8}$$

Letting $p = 1, \alpha = 1$ in Theorem 1, we have

COROLLARY 3. *If $f(z) \in A(n) = \mathcal{A}(1, n)$ with $Ref'(z) > \beta$, then*

$$Re\left(\frac{f(z)}{z}\right) > \frac{n + 2\beta}{n + 2} \ (z \in U). \tag{2. 9}$$

REMARK. *If we take $\beta = 0$ in Corollary 3, we have the Theorem 2 by Cho([1]).*

Using the above Lemma 2, we have the following

THEOREM 2. *Let $f(z) \in \mathcal{B}_1(p,n,\alpha,\beta)$ with $\alpha > 0$ and $0 \le \beta < p$, then*

$$Re\left(\frac{f(z)}{z^p}\right)^{\frac{\alpha}{2}} > \frac{n+\sqrt{n^2+4p\alpha\beta(n+p\alpha)}}{2(n+p\alpha)} \quad (z \in U). \tag{2. 10}$$

*Proof.* We define the function $h(z)^p = f(z)$ for $f(z) \in \mathcal{B}_1(p,n,\alpha,\beta)$ as Theorem 1. Then we have

$$h(z) \in \mathcal{B}_1(1,n,p\alpha,\frac{\beta}{p}).$$

By Lemma 2, we have

(2. 11)

$$\begin{aligned} Re\left(\frac{f(z)}{z^p}\right)^{\frac{\alpha}{2}} &= Re\left(\frac{h(z)}{z}\right)^{\frac{p\alpha}{2}} \\ &> \frac{n+\sqrt{n^2+4p\alpha\beta(n+p\alpha)}}{2(n+p\alpha)}. \end{aligned}$$

Letting $n = 1$ in Theorem 2, we have

COROLLARY 4. *Let $f(z) \in \mathcal{B}_1(p,1,\alpha,\beta)$ with $\alpha > 0$ and $0 \le \beta < p$, then*

$$Re\left(\frac{f(z)}{z^p}\right)^{\frac{\alpha}{2}} > \frac{1+\sqrt{1+4\alpha\beta(1+p\alpha)}}{2+2p\alpha} \quad (z \in U). \tag{2. 12}$$

Letting $p = 1$ in Theorem 2, we have

COROLLARY 5([3]). *If $f(z) \in B(n,\alpha,\beta) = \mathcal{B}_1(1,n,\alpha,\beta)$ with $\alpha > 0$ and $0 \le \beta < 1$, then*

$$Re\left(\frac{f(z)}{z^p}\right)^{\frac{\alpha}{2}} > \frac{n+\sqrt{n^2+4\alpha\beta(n+p\alpha)}}{2(n+\alpha)} \quad (z \in U). \tag{2. 13}$$

If we take $p = 1, \alpha = 2$ in Theorem 2, we have

COROLLARY 6([1]). *If* $f(z) \in B(n,2) = \mathcal{B}_(1,n,2,\beta)$, *then*

(2. 14)
$$\frac{f(z)}{z} > \frac{n+\sqrt{n^2+8\beta(n+\alpha)}}{2(n+2)}.$$

REMARK. *If we take* $\beta = 0$ *in Corollary 6, we have Theorem 3 due to Cho([1]).*

Theorem 3 Let $f(z) \in \mathcal{B}_1(p,n,\alpha,\beta)$ with $\alpha$ and $0 \le \beta < p$ and $G(z)$ defined by

(2. 15)
$$G(Z) = (z^{p\gamma} f(z)^{\alpha})^{\frac{1}{\alpha+\gamma}} \ (\gamma \ge 0).$$

Then $G(z)$ is in the class $\mathcal{B}_1(p,n,\alpha+\gamma,\delta)$, where

(2. 16)
$$\delta = \frac{1}{\alpha+\gamma}\left(\frac{p\gamma(n+2\alpha\beta)}{n+2p\alpha}+\alpha\beta\right).$$

*Proof.* Differenting both sides of (2. 15) we have

(2. 17)
$$(\alpha+\gamma)G'(z)G(z)^{(\alpha+\gamma)-1} = p\gamma z^{p\gamma-1} f(z)^{\alpha} + \alpha z^{p\gamma} f'(z) f(z)^{\alpha-1}.$$

By Theorem 1 and (2. 17), we have

$$\begin{aligned}
&Re\left(\frac{zG'(z)G(z)^{(\alpha+\gamma)-1}}{z^{p(\alpha+\gamma)}}\right)\\
&= \frac{1}{\alpha+\gamma}\left(p\gamma Re\left(\frac{f(z)}{z^p}\right) + \alpha Re\left(\frac{zf'(z)f(z)^{\alpha-1}}{z^{p\alpha}}\right)\right)\\
&\ge \frac{1}{\alpha+\gamma}\left(\frac{p\gamma(n+2\alpha\beta)}{n+2p\alpha}+\alpha\beta\right).
\end{aligned}$$

Letting $n = 1$ in Theorem 3, we have

COROLLARY 7([3]). *Let $f(z) \in B_1(p, \alpha, \beta) = \mathcal{B}_1(p, n, \alpha, \beta)$ with $\alpha$ and $0 \leq \beta < p$ and $G(z)$ defined by*

$$G_1(Z)^{\alpha+\gamma} = z^{p\gamma} f(z)^{\alpha} \ (\gamma \geq 0). \tag{2. 18}$$

*is in the class $B_1(p, \alpha + \gamma) = \mathcal{B}_1(p, n, \alpha + \gamma, \delta)$,* where

$$\delta = \frac{1}{\alpha + \gamma} \left( \frac{p\gamma(n + 2\alpha\beta)}{1 + 2p\alpha} + \alpha\beta \right). \tag{2. 19}$$

THEOREM 4. *Let $f(z) \in \mathcal{B}_(p, n, \alpha\beta)$ with $\alpha > 0$ and $0 \leq \beta < p$ and $H(z)$ defined by*

$$H(z) = \left( z^{p\gamma} f(z)^{\frac{\alpha}{2}} \right)^{\frac{2}{\alpha+2\gamma}}. \tag{2. 20}$$

*Then $H(z)$ is in the class $B_1(p, n, \frac{\alpha}{2} + \gamma, \delta)$,* where

$$\delta = \frac{1}{\alpha + 2\gamma} \left( \frac{p\gamma(n + \sqrt{n^2 + 4\alpha\beta(n + \beta\alpha)})}{n + p\alpha} + 2\alpha\beta \right). \tag{2.21}$$

*Proof.* Differenting both side of (2. 20), we have

$$(\frac{\alpha}{2} + \gamma) H'(z) H(z)^{\frac{\alpha}{2}+\gamma-1} = p\gamma z^{p\gamma-1} f(z)^{\frac{\alpha}{2}} + \frac{\alpha}{2} z^{p\gamma} f'(z) f(z)^{\frac{\alpha}{2}-1},$$

or

$$\frac{zH'(z)h(z)^{\frac{\alpha}{2}+\gamma-1}}{z^{p(\frac{\alpha}{2}+\gamma)}} = \frac{2p\gamma}{\alpha + 2\gamma} \left( \frac{f(z)}{z^p} \right)^{\frac{\alpha}{2}} + \frac{2\alpha}{\alpha + 2\gamma} \left( \frac{zf'(z)f(z)^{\frac{\alpha}{2}-1}}{z^{p\frac{\alpha}{2}}} \right).$$

By Theorem 2, we have

$$Re \left( \frac{zH'(z)H(z)^{\frac{\alpha}{2}+\gamma-1}}{z^{p(\frac{\alpha}{2}+\gamma)}} \right) \geq \frac{1}{\alpha+2\gamma} \left( \frac{p\gamma(n+\sqrt{n^2+4\alpha\beta(n+\beta\alpha)})}{n+p\alpha} + 2\alpha\beta \right).$$

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