

ON LINE GRAPHS OF CYCLIC COVERING GRAPHS

MOO YOUNG SOHN * AND HYE KYUNG KIM†

1. Introduction

Let G be a connected finite simple graph with vertex set $V(G)$ and edge set $E(G)$. The *neighborhood* of a vertex $v \in V(G)$, denoted by $N(v)$, is the set of vertices adjacent to v . We use $|X|$ for the cardinality of a set X . A graph G is called a (p, q) -graph if $|V(G)| = p$ and $|E(G)| = q$.

The line graph $L(G)$ of G is the graph whose vertices are the edges of G , with two vertices of $L(G)$ are adjacent whenever the corresponding edges of G share a common vertex. Therefore, if G is a (p, q) -graph, then the line graph $L(G)$ has q vertices and $\frac{1}{2} \sum_{v \in V(G)} d(v)^2 - q$ edges, where $d(v)$ is the degree of v .

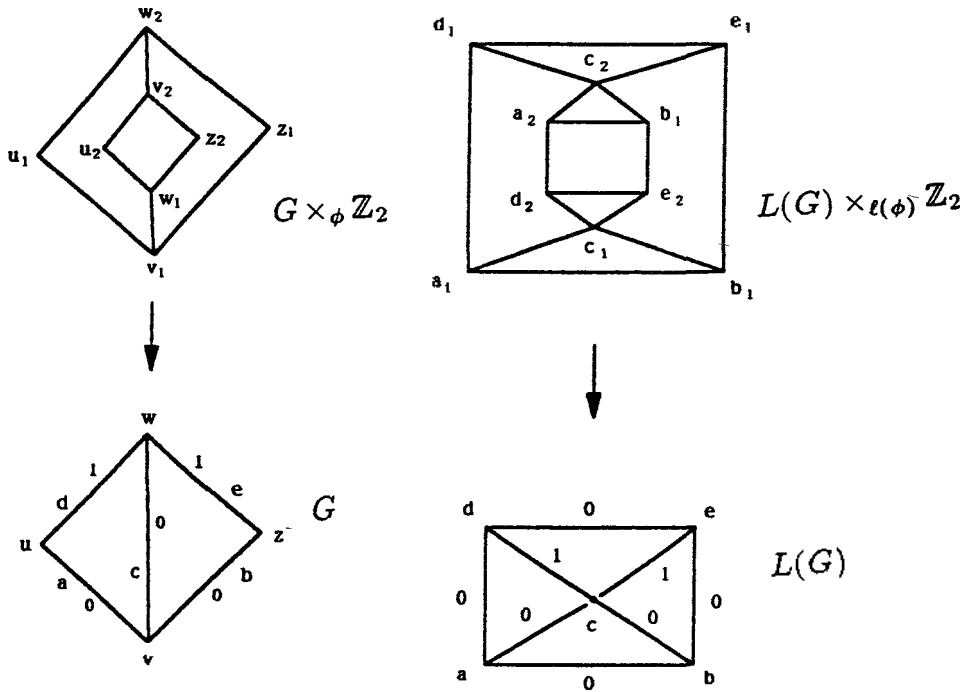
A graph \tilde{G} is called a *covering* of G with projection $p : \tilde{G} \rightarrow G$ if there is a surjection $p : V(\tilde{G}) \rightarrow V(G)$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \rightarrow N(v)$ is a bijection for any vertex $v \in V(G)$ and $\tilde{v} \in p^{-1}(v)$. We also say that the projection $p : \tilde{G} \rightarrow G$ is an *n-fold covering* of G if p is n -to-one. A covering $p : \tilde{G} \rightarrow G$ is said to be *regular* (simply, *\mathcal{A} -covering*) if there is a subgroup \mathcal{A} of the automorphism group $\text{Aut}(\tilde{G})$ of \tilde{G} acting freely on \tilde{G} so that the graph G is isomorphic to the quotient graph \tilde{G}/\mathcal{A} , say by h , and the quotient map $\tilde{G} \rightarrow \tilde{G}/\mathcal{A}$ is the composition $h \circ p$ of p and h . If the covering transformation group \mathcal{A} of the covering \tilde{G} is the cyclic group \mathbb{Z}_n , then we call \tilde{G} a *cyclic covering* of G . When we wish to specify n , we call it *n-fold cyclic covering graph*. The fibre of an edge or a vertex is its preimage under p . Every \mathcal{A} -covering of a graph G can be constructed as follows [2]:

Convert G to a digraph \vec{G} by replacing each edge e of G with a pair of oppositely directed edges. By $e^{-1} = vu$, we mean the reverse edge

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to a directed edge $e = uv$. We denote the set of directed edges of \vec{G} by $E(\vec{G})$. For an edge $e = uv \in E(\vec{G})$, we denote by i_e and t_e the initial vertex u and the terminal vertex v of e , respectively. An \mathcal{A} -voltage assignment of G is a function $\phi : E(\vec{G}) \rightarrow \mathcal{A}$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in E(\vec{G})$. We denote $C^1(G; \mathcal{A})$ the set of all \mathcal{A} -voltage assignments of G . The derived graph $G \times_\phi \mathcal{A}$ derived from an \mathcal{A} -voltage assignment $\phi : E(\vec{G}) \rightarrow \mathcal{A}$ has as its vertex set $V(G) \times \mathcal{A}$ and as its edge set $E(G) \times \mathcal{A}$, so that an edge of $G \times_\phi \mathcal{A}$ joins a vertex (u, g) to $(v, \phi(e)g)$ for $e = uv \in E(\vec{G})$ and $g \in \mathcal{A}$. In the derived graph $G \times_\phi \mathcal{A}$, a vertex (u, g) is denoted by u_g , and an edge (e, g) by e_g . Define an \mathcal{A} -action Φ on $G \times_\phi \mathcal{A}$ by $\Phi(g)(v_{g_1}) = v_{g_1 g^{-1}}$ for each $v \in V(G)$, and $\Phi(g)(e_{g_1}) = e_{g_1 g^{-1}}$ for each $e \in E(\vec{G})$. Clearly, the first coordinate projection $p_\phi : G \times_\phi \mathcal{A} \rightarrow G$ is an \mathcal{A} -covering projection. In this paper, we show that the line graph of \mathbb{Z}_n -covering graph of a graph G can be expressed as a \mathbb{Z}_n -covering graph of the line graph of G .



< Fig. 1 >

2. Main Theorem

Let $p : \tilde{G} \rightarrow G$ be a cyclic covering graph. Then there is a \mathbb{Z}_n -voltage assignment ϕ of G such that $p : \tilde{G} \rightarrow G$ is isomorphic to $p_\phi : G \times_\phi \mathbb{Z}_n \rightarrow G$ as coverings of G , i.e., there exists a graph isomorphism $\Phi : \tilde{G} \rightarrow G \times_\phi \mathbb{Z}_n$ such that $p_\phi \Phi = p$.

Now, we aim to show that the line graph $L(G \times_\phi \mathbb{Z}_n)$ of the cyclic covering $G \times_\phi \mathbb{Z}_n$ can be constructed as a cyclic covering of the line graph $L(G)$ of G .

Let ϕ be a voltage assignment in $C^1(G; \mathbb{Z}_n)$. Define $\ell(\phi) : D(L(G)) \rightarrow \mathbb{Z}_n$ as follows. Let $ef \in D(L(G))$. Then $e, f \in E(G)$.

$$p^{-1}(e) = \{e_0, e_1, e_2, \dots, e_{n-1}\} \quad \text{and} \quad p^{-1}(f) = \{f_0, f_1, f_2, \dots, f_{n-1}\}.$$

We consider following three cases.

Case(i). If $e^+ = uv$, $f^+ = uw$, $\phi(e) = a$ and $\phi(f) = b$, then there exist an edge which joins the vertex $e_i = u_i v_{a+i}$, and the vertex $f_i = u_i w_{b+i}$, for all $i \in \mathbb{Z}_n$, in the line graph $L(G \times_\phi \mathbb{Z}_n)$ of $G \times_\phi \mathbb{Z}_n$. Define $\ell(\phi)(ef) = 0$, where 0 is the identity in \mathbb{Z}_n .

Case(ii). If $e^+ = uv$, $f^+ = vw$, $\phi(e) = a$ and $\phi(f) = b$, then there exist an edge which joins the vertex $e_i = u_i v_{a+i}$, and the vertex $f_{a+i} = v_{a+i} w_{b+a+i}$, for all $i \in \mathbb{Z}_n$, in the line graph $L(G \times_\phi \mathbb{Z}_n)$ of $G \times_\phi \mathbb{Z}_n$. Define $\ell(\phi)(ef) = a$.

Case(iii). If $e^+ = uv$, $f^+ = wv$, $\phi(e) = a$ and $\phi(f) = b$, then there exist an edge which joins the vertex $e_i = u_i v_{a+i}$, and the vertex $f_{-b+a+i} = w_{-b+a+i} v_{a+i}$, for all $i \in \mathbb{Z}_n$, in the line graph $L(G \times_\phi \mathbb{Z}_n)$ of $G \times_\phi \mathbb{Z}_n$. Define $\ell(\phi)(ef) = -b + a$.

By the construction of $\ell(\phi)$, we can see that $L(G) \times_{\ell(\phi)} \mathbb{Z}_n$ is isomorphic to the line graph $L(G \times_\phi \mathbb{Z}_n)$ of $G \times_\phi \mathbb{Z}_n$. We illustrate this in the figure 1. Now we summarize our discussion as follows.

THEOREM 1. *Let $p : \tilde{G} \rightarrow G$ be a cyclic covering graph of a simple connected graph G . Then the line graph $L(\tilde{G})$ of the covering graph \tilde{G} can be represented as a cyclic covering graph of the line graph $L(G)$.*

3. Applications

The adjacency matrix $A(G) = (a_{ij})$ of G is the square matrix of order $|V(G)|$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } i \neq j, e = v_i v_j, \\ 0 & \text{if } i = j. \end{cases}$$

Then the characteristic polynomial $\Phi(G; \lambda)$ of G is the characteristic polynomial $\det(\lambda I - A(G))$ of $A(G)$.

Let \mathbb{C} denote the field of complex numbers, and let D be a digraph. A *weighted digraph* is a pair $D_\omega = (D, \omega)$, where $\omega : E(D) \cup V(D) \rightarrow \mathbb{C}$ is a function on the set $E(D) \cup V(D)$ of edges and vertices in D . We call D the *underlying digraph* of D_ω and ω the *weight function* of D_ω .

Given any weighted digraph D_ω , the adjacency matrix $A(D_\omega) = (a_{ij})$ of D_ω is the square matrix of order $|V(D)|$ defined by

$$a_{ij} = \begin{cases} \omega(v_i v_j) & \text{if } v_i v_j \in E(D), \\ \omega(v_i) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

and its characteristic polynomial is that of its adjacency matrix. We shall denote the characteristic polynomial of D_ω by $\Phi(D_\omega; \lambda)$.

In this section we shall determine the characteristic polynomial of the line graphs of cyclic covering graphs. In the proofs of several theorems we shall see the following lemma from the general theory of matrices [1].

LEMMA 1. *Let $\Phi(X; \lambda)$ denote the characteristic polynomial of the square matrix X . If A is an $p \times q$ -matrix, then*

$$\lambda^q \Phi(AA^t; \lambda) = \lambda^p \Phi(A^t A; \lambda).$$

Let $R(G)$ be the $p \times q$ -incident matrix of vertices and edges of the graph G and $D(G)$ the degree matrix of G . Then

$$R(G)R(G)^t = A(G) + D(G) \quad \text{and} \quad R(G)^t R(G) = A(L(G)) + 2I.$$

By using the lemma 1, we can see that

$$|(\lambda - 2)I - A(L(G))| = |\lambda I - R(G)^t R(G)| = \lambda^{q-p} |\lambda I - A(G) - D(G)|.$$

It implies that

$$\Phi(L(G); \lambda - 2) = \lambda^{q-p} |\lambda I - A(G) - D(G)|,$$

and hence

$$\Phi(L(G); \lambda) = (\lambda + 2)^{q-p} |(\lambda + 2)I - A(G) - D(G)|$$

In particular, If G is a regular graphs of degree r with p vertices, then we have

$$\Phi(L(G); \lambda) = (\lambda + 2)^{q-p} \Phi(G; \lambda - r + 2).$$

Now, for any \mathbb{Z}_n -voltage assignment ϕ of G and each $\gamma \in \text{Aut}(\mathbb{Z}_n)$, let $\vec{G}_{(\phi, \gamma)}$ denote the spanning subgraph of the digraph \vec{G} whose directed edge set of $\phi^{-1}(\gamma)$, so that the digraph \vec{G} is the edge-disjoint union of spanning subgraphs $\vec{G}_{(\phi, \gamma)}$, $\gamma \in \text{Aut}(\mathbb{Z}_n)$. Let $\omega_i(\phi) : E(\vec{G}) \rightarrow \mathbb{C}$ be the function defined by $\omega_i(\phi)(e) = \gamma(\phi(e), i)$ for $e \in E(\vec{G})$, so that the adjacency matrix of a weighted digraph $(\vec{G}, \omega_i(\phi))$ is the matrix

$$\sum_{\gamma \in \text{Aut}(\mathbb{Z}_n)} \lambda_{(\gamma, i)} A(\vec{G}_{(\phi, \gamma)})$$

for each $i = 1, 2, \dots, n$ and each $\lambda_{(\gamma, i)}$ is an eigenvalue of the permutation matrix $P(\gamma)$. By using these facts, Theorem 1 and Theorem 3 in [3], we have

THEOREM 2. *Let G be a simple connected graph and ϕ be an voltage assignment in $C^1(G; \mathbb{Z}_n)$. Then we have*

$$\Phi(L(G \times_{\phi} \mathbb{Z}_n); \lambda) = \prod_{j=0}^{n-1} \Phi(\vec{L(G)}_{\omega_j(\phi)} : \lambda),$$

where $\omega_j(\phi)(e) = \begin{cases} \exp(\frac{2\pi i}{n})^{jk} & \text{if } \ell(\phi)(e) = k, \\ 0 & \text{otherwise} \end{cases}$, for each j ,

and

$$\Phi(L(G \times_{\phi} \mathbb{Z}_n); \lambda) = (\lambda + 2)^{n(q-p)} \prod_{j=0}^{n-1} \Phi(\vec{G}_{\omega_j(\phi)} : \lambda + 2),$$

where $\omega_j(\phi)(e) = \begin{cases} \exp(\frac{2\pi i}{n})^{jk} & \text{if } \phi(e) = k, \\ 0 & \text{otherwise} \end{cases}$ for each i and $\omega_j(\phi)(v_s) = d(v_s)$ for each j and s .

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Department of Mathematics
Changwon National University
Changwon, 641–773 Korea

Department of Mathematics
Chatholic University of Taegu Hyosung
Kyongsan, 713–702 Korea