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A STUDY ON DERIVATIONS IN NEAR–RINGS

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1. Introduction

Thoughout, we will consider that N is a zero-symmetric near-ring with multiplicative center $Z(N)$. A derivation on N is an additive endomorphism D satisfying the product rule

$$D(xy) = D(x)y + xD(y)$$

for all x and y in N.

Every element x of N such that $D(x) = 0$ is called constant. For x and y in N the symbol $[x, y]$ will denote the commutator $xy - yx$, while the symbol (x, y) will denote the additive group commutator $x + y - x - y$ and the symbol $x \circ y$ will denote the skew commutator $xy + yx$.

A derivation D is called centralizing if $[D(x), x] \in Z(N)$ for all X in N, skew centralizing if $D(x) \circ X \in Z(N)$ for all x in N, in particular, D is called commuting if $[D(x), x] = 0$ for all x in N, skew commuting if $D(x) \circ x = 0$ for all x in N and an element x of N with $[D(x), x] = 0$ is called commuting.

In ring theory, many mathematicians, H.E.Bell and G.Mason [1], M.Bresar [2], [3] in 1993, I.N.Herstain [8], [9] and E.C.Posner [11], studied centralizing and commuting mappings, endomorphisms, automorphisms or nil and nilpotent derivations of prime rings,or semiprime rings. Some of them has derived commutativity. We shall investigate some characterizations of near-ring with a derivation and derive that any prime near-ring with a derivation and certain conditions becomes a commutative ring. In order to prove our theorem, we will introduce several useful lemmas.

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We will consider nilpotent and nil derivations on N. In ring theory these concepts are studied by L.O.Chung and J.Luh [5], [6] in 1984, [4] in 1985 and P.Grzesezczuk [7] in 1992. We shall prove that for a left strongly prime near-ring, in particular, prime near-ring with DCC on left annihilators, every nil derivation N on a non-zero left ideal of N is also nil on N.

For any subset S of N, we write $Z(S)$ the center of S and A is denoted by an additeve non-zero N-subgroup of N. We now introduce the following special near-rings which are well known in [10] : A near-ring N is reduced if it has no non-zero nilpotent elements, prime if $a, b \in N$ and $aNb = 0$ implies $a = 0$ or $b = 0$ and has the insertion of factors property (abbr. IFP) provided that $ab = 0$ implies $axb = 0$ for all x in N.

2. Properties of Near-Rings with Derivations

LEMMA 2.1[10]. *Let N be any near-ring.*

(1) *If N is reduced, then N has the IFP.*

(2) *N has the IFP if and only if for any a in N, $(0 : a)$ is ideal of N if and only if for any $S \subset N, (0 : S)$ is an ideal of N.*

A derivation D on N is said to be nilpotent if there is a positive integer n such that $D^n(a) = 0$ for all a in N. The least such number is called the index of nilpotency of D, denoted by $nil(D)$.D is said to be nil if for each a in N, there is a natural number n (depending on a) such that $D^n(a) = 0$. The least such number is called the index of nilpotency of D with respect to a, denoted by $nilD, a)$.) Obviously nilpotent derivation is nil, but not vice versa. The latter can be seen from the following example : Consider, the near-ring $R[x]$ of polynomials over a commutative ring. Let D be the ordinary derivation : $D(x^n) = nx^{n-1}$. It is routine to see that D is nil but not nilpotent.

THEOREM 2.2. *Let D be a derivation on a reduced near-ring N. Then every annihilator ideal is invariant under D. Moreover, we have an ascending chain of annihilator ideals.*

Proof. Let S be any subset of N. Consider $(0 : S)$ as annihilator ideal of N with respect to S. We must show that

$$D(0 : S) \subset (0 : S).$$

In fact, let $x \in (0 : S)$, that is, $xs = 0$ for all s in S. Taking D, we have

$$0 = D(xs) = D(x)s + xD(s).$$

Multiplying s to the right side of this equality, we get $0 = D(x)s^2 + xD(s)s$. Since N is reduced by Lemma 2.1. (1), N has the IFP, so that $xD(S)s = 0$. Hence we get $D(x)s^2 = 0$ for all s in S. Again multiplying $D(x)$ to the left side of $D(x)s^2 = 0, (D(x)s)^2 = 0$. Because N is reduced we see that $D(x)s = 0$ for all s in S. Therefore $D(x) \in (0 : S)$. Moreover, let $xs = 0$ for all s in S. Taking derivation $D, 0 = D(xs) = D(x)s + xD(s)$. Multiplying x to the left side of this equality, $0 = xD(x)s + x^2D(s)$. Since N is reduced, from Lemma 2.1, we obtain that

$$xD(x)s = 0 \; and \; x^2D(s) = 0.$$

From the latter equality, we get $xD(s) = 0$ for all s in S by using reducidility and multiplying $D(s)$ to the right side. Consequently we obtain $xD(S) = 0$. Hence $x \in (0 : D(S))$. The proof of the last statement is that

$$(0 : S) \subset (0 : D(S)).$$

Next we put $S' = D(S)$ and again, taking the above procedure to S' we obtain the inclusion $(0 : S') \subset (0 : D(S'))$, that is to say,

$$(0 : D(S)) \subset (0 : D^2(S))$$

If we repeat this procedure continuously, we have the following ascending chain of annihilator ideals of N :

$$(0 : S) \subset (0 : D(S)) \subset (0 : D^2(D)) \subset ......$$

In particular for any a in N we obtain that

$$(0 : a) \subset (0 : D(a)) \subset (0 : D^2(a)) \subset ......$$

as ascending chain of principal annihilator ideals of N by using D repeatedly. □

From the Theorem 2.2, we have the following property of nil derivations.

Corollary 2.3. *Let D be a nil derivation on a reduced near-ring N. Then N has the ACC on principal annihilator ideals of N by applying D.*

Lemma 2.4. *Let D be a derivation on N. Then N satisfies the following left distributive law*

$$a(D(bc)) = a(D(b)c + bD(c)) = aD(b)c + abD(c)$$

for all a, b, c in N.

Proof. From the expression for $D((ab)c) = D(a(bc))$, we have our result. □

Lemma 2.5. *Let D be any derivation on N. If a in A is not a right zero divisor and a is commuting or skew commuting element, then for any element b in N (a, b) is constant.*

Proof. From the equality $(a+b)a = a^2 + ba$, we obtain

$$D(a)a + D(b)a + aD(a) + bD(a) = D(a)a + aD(a) + D(b)a + bD(a).$$

At first, for a is commuting this equation reduces to

$$D(a) + D(b) - D(a) - D(b)a = 0,$$

that is, $D(a+b-a-b)a = 0$. Since a is not a right zero divisor,we have

$$D(a,b) = D(a+b-a-b) = 0.$$

Secondly, for a is skew commuting, that is, $D(a)a = -aD(a)$, above equation reduces to

$$D(b)a - D(a)a = -D(a)a + D(b)a,$$

in other words,$D(a+b-a-b)a = 0$. Again, because a is not a right zero divisor

$$S(a,b) = D(a+b-a-b) = 0.$$

Therefore, in both the above cases, we see that (a, b) is constant. □

THEOREM 2.6. *Suppose a near-ring N has no non-zero divisors of zero. If N has a nontrivial commuting or skew commuting derivation D, then N is an abelian near-ring.*

Proof. Let c be any additive commutator in N. Then c is constant by Lemma 2.4 and 2.5. Moreover, for any x in N, cx is also an additive commutator, so that cx is also constant. Thus we obtain that

$$0 = D(cx) = D(c)x + cD(x) \text{ and } cD(x) = 0.$$

Since D is nontrivial, there exists y in N such that $D(y) \neq 0$. On the other hand we see that $cD(y) = 0$ as above case. Since N has no non-zero divisors of zero, we conclude that $c = 0$. Hence N is abelian. □

THEOREM 2.7. *Let N be a prime near-ring with IFP.*

(1) *If $(A, +)$ is abelian, then N is an abelian near-ring.*
(2) *If $(A, \cdot)$ is commutative, then N is a commutative near-ring.*

Proof. (1) Let x, y be in N and a in A. Since N is an additive abelian N-subgroup, we have the following equality :

$$xa + xa + ya + ya = xa + ya + xa + ya$$

If we take the left and right cancellation, this equality becomes

$$(x, y)a = (x + y - x - y)a = 0.$$

From the fact that N has the IFP, we have $(x, y)Na = 0$. If we take $a \in A$ as a non-zero element and use the fact that N is prime, then $(x, y) = 0$. Whence N is an abelian near-ring.
(2) Let x, y be any elements in N and take a and b are both non-zero elements of A. Since $(A, \cdot)$ is commutative, we obtain the following equalities:

$$[x, y]ab = xyab - yxab = xyab - ybxa = xayb - ybxa = ybxa - ybxa = 0.$$

From the fact N has the IFP, $[x, y]aNb = 0$. Applying N is prime, then $[x, y]a = 0$. Again from the fact that N has the IFP, we get $[x, y]Na = 0$, and again applying the property N is prime, consequently, we see that $[x, y] = 0$. Therefore N is commutative. □

In Theorem 2.7, We note that for any prime near-ring N with IFP, if A is a non-zero additive abelian commutative N-subgroup, then N is a commutative ring.

THEOREM 2.8. *Let N be any near-ring with derivation D such that $D^2 \neq 0$. Then every subnear-ring B generated by $D(N)$ contains a non-zero two sided N-subgroup of N.*

Proof. Since $D^2 \neq 0$ and $D(N)$ is contained in B, we have that $D^2(B) \neq 0$. Pick y in B such that $D^2(y) \neq 0$. If $x \in N$, then $D(x)y + xD(y) = D(xy)$ is contained in B, and since both y and $D(x)$ are in B we know that $xD(y) \in B$, which is to say, $ND(y) \subset B$. Similarly, we have $D(y)N \subset B$. Let a, b be in N. Then

$$D(a)D(y)b + aD^2(y)b + aD(y)D(b) = DaD(y)b$$

is contained in B. But by the above fact, $D(y)b \in B$ and $aD(y) \in B$. Consequently, we obtain that $aD^2(y)b \in B$ for all a, b in N. Furthermore, by the above fact,

$$ND^2(y)b \subset B \text{ and } D(a)D(y) + aD^2(y) = DaD(y) \in B.$$

Similarly,

$$D^2(y)N \subset B \text{ and } ND^2(y)N \subset B.$$

Hence B contains a non-zero two sided N-subgroup of N generated by $D^2(y) \neq 0$. Our proof is complete. □

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