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ON SELF MAPS OF HILBERT ALGEBRAS

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1. Introduction

In 1966, Diego [6] introduced the notions of Hilbert algebras and deductive systems, and proved various properties. The theory of Hilbert algebras and deductive systems was further developed by Busneag in [2-4]. In 1980, Schmid [10] defined the concept of maximal lattice of quotients for a distributive lattice. Using Schmid's idea, Busneag [3] constructed the notions of Hilbert algebra of fraction and maximal Hilbert algebra of quotients for a Hilbert algebra, and proved the existence of the maximal Hilbert algebra of quotients for a Hilbert algebra. For the general development of Hilbert algebras the deductive system theory plays an important role. Jun [9] gave some characterizations of deductive systems. In this paper we introduce the concept of right maps in Hilbert algebras. We study the properties of Hilbert algebras in the language of right maps. We show that the set of all right maps of a Hilbert algebra H is also a Hilbert algebra, and give a characterization of a deductive system by using right maps.

2. Preliminaries

We review some definitions and well known results.

DEFINITION 2.1 (DIEGO [6]). *A Hilbert algebra is a triple* $(H, \to, 1)$, *where* H *is a nonempty set,* $\to$ *is a binary operation on* H, $1 \in H$ *is an element such that the following three axioms are satisfied for every* $x, y, z \in H$:

(i) $x \to (y \to x) = 1$,
(ii) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,
(iii) *If* $x \to y = y \to x = 1$ *then* $x = y$.

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If H is a Hilbert algebra, then the relation $x \leq y$ iff $x \to y = 1$ is a partial order on H, which will be called the natural ordering on H. With respect to this ordering, 1 is the largest element of H.

PROPOSITION 2.2 (BUSNEAG [3] AND DIEGO [6]). *If H is a Hilbert algebra and $x, y, z \in H$, then the following hold:*

(1) $x \leq y \to x$.
(2) $x \to 1 = 1$.
(3) $x \to (y \to z) = (x \to y) \to (x \to z)$.
(4) $(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)$.
(5) $x \to (y \to z) = y \to (x \to z)$.
(6) $x \leq (x \to y) \to y$.
(7) $((x \to y) \to y) \to y = x \to y$.
(8) $1 \to x = x$.
(9) $x \to y \leq (y \to z) \to (x \to z)$.
(10) *If $x \leq y$, then $z \to x \leq z \to y$ and $y \to z \leq x \to z$.*

DEFINITION 2.3 (BUSNEAG [2]). *If H is a Hilbert algebra, a subset D of H is called a deductive system of H if it satisfies:*

(i) $1 \in D$,
(ii) *If $x, x \to y \in D$, then $y \in D$.*

3. Self maps of Hilbert algebras

We begin this section by giving the definition of right maps in Hilbert algebras.

DEFINITION 3.1. *Let H be a Hilbert algebra and $x \in H$. A mapping $R_x : H \to H$ defined by $R_x(t) = t \to x$ for all $t \in H$ is called a right map of H, and denote $\mathcal{R} = \{R_x : x \in H\}$ and $R_x^n = R_x \circ \ldots \circ R_x$ (R_x appears n times).*

By using the notion of right maps, we can directly express some fundamental properties of a Hilbert algebra H as follows:

(1) $x \leq R_x(y)$.
(2) $R_1(x) = 1$ and $R_x(1) = x$.
(3) $R_{R_x(y)}(z) = R_{R_x(z)}(y) = R_{R_x(z)}(R_y(z))$.
(4) $R_{R_x^2(y)}(R_y(x)) = R_{R_y^2(x)}(R_x(y))$.

(5) $x \le R_y^2(x)$.
(6) $R_y^3(x) = R_y(x)$.
(7) $R_y(x) \le R_{R_z(x)}(R_z(y))$.

For a right map R_x we have the following properties.

THEOREM 3.2. *Let H be a Hilbert algebra and $x \in H$. Then*
(1) *If R_x is injective then x is a minimal element of H.*
(2) *If R_x is surjective then x is the least element of H.*
(3) *If R_x is surjective then it is injective.*

Proof. (1) Let R_x be injective and suppose $y \le x$ for some $y \in H$. Then $x \to x = 1 = y \to x$, which implies $R_x(x) = R_x(y)$. Since R_x is injective, therefore $x = y$. This means that x is a minimal element of H.

(2) Let y be an arbitrary element of H. As R_x is surjective, there exists an element $t \in H$ such that $t \to x = y$. It follows from (2) and (5) of Proposition 2.2 that

$$x \to y = x \to (t \to x) = t \to (x \to x) = t \to 1 = 1.$$

This shows that x is the least element of H.

(3) Suppose that R_x is surjective and $R_x(s) = R_x(t)$ for $s, t \in H$. By definition of R_x we have $s \to x = t \to x$. As R_x is surjective, for the element t there exists an element $y \in H$ such that $y \to x = t$. It follows from Proposition 2.2(6) that $y \le (y \to x) \to x = t \to x = s \to x$ so that, by using Proposition 2.2(5), $1 = y \to (s \to x) = s \to (y \to x) = s \to t$. Similarly we obtain $t \to s = 1$. Thus $s = t$ and R_x is injective. □

To prove the converse of Theorem 3.2(3), we need the following lemma.

LEMMA 3.3. *Let H be a Hilbert algebra, $x \in H$ and R_x a right map of H. Then R_x is injective if and only if $(u \to x) \to x = u$ for every $u \in H$.*

Proof. Let R_x be an injective map. By Proposition 2.2(7), we have

$$((u \to x) \to x) \to x = u \to x \text{ for every } u \in H.$$

This implies $R_x((u \to x) \to x) = R_x(u)$. Since R_x is injective, it follows that $(u \to x) \to x = u$ for every $u \in H$. Conversely suppose that $(u \to x) \to x = u$ for every $u \in H$ and let $R_x(s) = R_x(t)$. Then $s = (s \to x) \to x = (t \to x) \to x = t$, and so R_x is injective. □

By using Theorem 3.2(3) and Lemma 3.3, we know that the injectivity and the surjectivity of a right map are coincide as mentioned in the following theorem.

THEOREM 3.4. *Let H be a Hilbert algebra and $x \in H$. Then R_x is injective if and only if R_x is surjective.*

Proof. It suffices to show that if R_x is injective then it is surjective. Let R_x be an injective map. By Lemma 3.3 we have $(u \to x) \to x = u$ for every $u \in H$. It follows that $R_x(u \to x) = u$, which means that R_x is surjective. Thus the theorem is true. □

The following theorem shows that every self map of a Hilbert algebra is uniquely expressed in the language of a right map.

THEOREM 3.5. *Let f be a self-map of a Hilbert algebra H satisfying $v \to f(u) = u \to f(v)$ for every $u, v \in H$. Then there exists a unique element x in H such that $R_x = f$.*

Proof. Let $x = f(1)$. Then for every $u \in H$,

$$\begin{aligned} f(u) \to R_x(u) &= f(u) \to (u \to x) \\ &= f(u) \to (u \to f(1)) \\ &= f(u) \to (1 \to f(u)) \\ &= f(u) \to f(u) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} R_x(u) \to f(u) &= (u \to x) \to f(u) \\ &= (u \to f(1)) \to f(u) \\ &= (1 \to f(u)) \to f(u) \\ &= f(u) \to f(u) \\ &= 1. \end{aligned}$$

Hence $R_x(u) = f(u)$ or $R_x = f$. Uniqueness is evident. □

Define a binary operation $\overset{R}{\to}$ on $\mathcal{R}$ as follows:

$$(R_x \overset{R}{\to} R_y)(t) = R_x(t) \to R_y(t) \text{ for all } t \in H.$$

Define $R_x \leq_R R_y$ if and only if $R_x(t) \leq R_y(t)$ for all $t \in H$. Then $R_x \leq_R R_y$ and $R_y \leq_R R_x$ imply $R_x = R_y$.

The following lemma is useful in the sequel.

LEMMA 3.6. *In a Hilbert algebra, we have the following properties:*

(1) $x \leq y$ *implies* $R_x \leq_R R_y$.

(2) $R_x \overset{R}{\to} R_y = R_{x \to y}$.

Proof. (1) Assume that $x \leq y$ and let t be any element of a Hilbert algebra H. Using Proposition 2.2(10), then

$$R_x(t) = t \to x \leq t \to y = R_y(t),$$

which means that $R_x \leq_R R_y$.

(2) For every $t \in H$, we obtain

$$\begin{aligned}(R_x \overset{R}{\to} R_y)(t) &= R_x(t) \to R_y(t)\\ &= (t \to x) \to (t \to y)\\ &= t \to (x \to y) \text{ [by Proposition 2.2(3)]}\\ &= R_{x \to y}(t).\end{aligned}$$

Hence $R_x \overset{R}{\to} R_y = R_{x \to y}$. □

THEOREM 3.7. *Let H be a Hilbert algebra. Then $(\mathcal{R}, \overset{R}{\to}, R_1)$ is a Hilbert algebra.*

Proof. Let $R_x, R_y, R_z \in \mathcal{R}$ and let t be any element of H. Then

$$(R_x \overset{R}{\to} (R_y \overset{R}{\to} R_x))(t) = (R_x \overset{R}{\to} R_{y \to x})(t) = R_{x \to (y \to x)}(t) = R_1(t)$$

and

$$\begin{aligned}
&((R_x \xrightarrow{R} (R_y \xrightarrow{R} R_z)) \xrightarrow{R} ((R_x \xrightarrow{R} R_y) \xrightarrow{R} (R_x \xrightarrow{R} R_z)))(t)\\
&= (R_{x\to(y\to z)} \xrightarrow{R} R_{(x\to y)\to(x\to z)})(t)\\
&= R_{(x\to(y\to z))\to((x\to y)\to(x\to z))}(t)\\
&= R_1(t).
\end{aligned}$$

These show that $R_x \xrightarrow{R} (R_y \xrightarrow{R} R_x) = R_1$ and

$$(R_x \xrightarrow{R} (R_y \xrightarrow{R} R_z)) \xrightarrow{R} ((R_x \xrightarrow{R} R_y) \xrightarrow{R} (R_x \xrightarrow{R} R_z)) = R_1.$$

Finally if $R_x \xrightarrow{R} R_y = R_y \xrightarrow{R} R_x = R_1$, then $R_{x\to y} = R_{y\to x} = R_1$, which implies that

$$R_{x\to y}(t) = R_{y\to x}(t) = R_1(t) \text{ or } t \to (x \to y) = t \to (y \to x) = t \to 1.$$

It follows from (2) and (3) of Proposition 2.2 that

$$(t \to x) \to (t \to y) = (t \to y) \to (t \to x) = 1.$$

Thus the condition (iii) of Definition 2.1 assures us that $t \to x = t \to y$ or $R_x(t) = R_y(t)$, so that $R_x = R_y$. This completes the proof. □

THEOREM 3.8. *Let H be a Hilbert algebra and $D \subseteq H$. Then D is a deductive system of H if and only if $\mathcal{R}_D$ is a deductive system of $\mathcal{R}$, where $\mathcal{R}_D = \{R_x : x \in D\}$.*

Proof. ($\Rightarrow$) $1 \in D$ implies $R_1 \in \mathcal{R}_D$. Let $R_x \in \mathcal{R}_D$ and $R_y \in \mathcal{R}$ such that $R_x \xrightarrow{R} R_y \in \mathcal{R}_D$. Then $R_{x\to y} = R_x \xrightarrow{R} R_y \in \mathcal{R}_D$, and so $x \to y \in D$. Since $R_x \in \mathcal{R}_D$, therefore $x \in D$. It follows from Definition 2.3(ii) that $y \in D$, so that $R_y \in \mathcal{R}_D$. This proves that $\mathcal{R}_D$ is a deductive system of $\mathcal{R}$.

($\Leftarrow$) Assume that $\mathcal{R}_D$ is a deductive system of $\mathcal{R}$. Then $R_1 \in \mathcal{R}_D$. Thus $R_1 = R_d$ for some $d \in D$. It follows that $R_1(t) = R_d(t)$ for all $t \in H$ or $t \to 1 = t \to d$. Taking $t = 1$ and using Proposition 2.2(8), then $1 = d \in D$. Now let $x \in D$ and $x \to y \in D$. Then

$R_x \in \mathcal{R}_D$ and $R_x \stackrel{R}{\to} R_y = R_{x \to y} \in \mathcal{R}_D$. This implies by Definition 2.3(ii) that $R_y \in \mathcal{R}_D$, and so $R_y = R_u$ for some $u \in D$. It follows that $R_y(t) = R_u(t)$ for all $t \in H$ or $t \to y = t \to u$. Repalcing t by 1 and using Proposition 2.2(8), we have $y = u \in D$, which proves that D is a deductive system of H. The proof is complete. □

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