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MΓ–HOMOMORPHISMS AND MΓ–IDEALS OF MΓ–GROUPS

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The concept of MΓ-groups was first introduced by Satyanarayana ([6]) who used the term MΓ-modules. In 1987, Booth chose the term MΓ-group for the same of consistency with Pilz([5]). In [1],[2] and [3]. G.L.Booth and Groenewald obtained some interesting properties of the radicals of Γ-near-rings. But now we show that the usual isomorphism theorems for rings hold for MΓ-groups.

In this paper, the term "near-ring" will mean a right (distributive) near-ring. A Γ-near-ring is a triple $(M,+,\Gamma)$ where

(i) $(M,+)$ is a (not necessarily abelian) group;

(ii) Γ is a nonempty set of binary operations on M such that for each $\gamma \in \Gamma$, $(M,+,\gamma)$ is a near-ring;

(iii) $x\gamma(y\mu z) = (x\gamma y)\mu z$ for all $x,y,z \in M$ and $\gamma,\mu \in \Gamma$.

If M is a Γ-near-ring, the zerosymmetric part of M is the set $M_0 = \{x \in M : x\gamma 0 = 0 \text{ for all } \gamma \in \Gamma\}$. If $M_0 = M$, M is called zerosymmetric. Throughout this paper,M denotes a zerosymmetric Γ-near-ring.

Definition 1[1]. *Let $(G,+)$ be a group. If, for all $x,y \in M$, $\gamma,\mu \in \Gamma$ and $g \in G$, it holds*

(i) $x\gamma g \in G$,

(ii) $(x+y)\gamma g = x\gamma g + y\gamma g$,

(iii) $x\gamma(y\mu g) = (x\gamma y)\mu g$,

then G is called an $M\Gamma$-group.

Let G and G' be MΓ-groups. If $f : G \to G'$ is a group homomorphism such that, for all $x \in M$, $\gamma \in \Gamma$ and $g \in G$, $f(x\gamma g) = x\gamma f(g)$, then f is called an MΓ-homomorphism. If f is bijective as well, then it is called an MΓ-isomorphism.

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If G is an MΓ-group, a subset H of G is called an MΓ-ideal of G if

(i) $(H, +)$ is a normal divisor of $(G, +)$, and

(ii) for all $x \in M, \gamma \in \Gamma, h \in H$ and $g \in G$, $x\gamma(g+h) - x\gamma g \in H$.

A subgroup K of G is called an MΓ-subgroup of G if $x\gamma k \in K$, for all $x \in M, \gamma \in \Gamma$ and $k \in K$. Since we are considering zerosymmetric Γ-near-rings only, every MΓ-ideal of G is also an MΓ-subgroup of G.

THEOREM 2. *Let $f : G \to G'$ be an MΓ-homomorphism of MΓ-groups. For any MΓ-subgroup H of G, $f(H) = \{f(a) | a \in H\}$ is an MΓ-subgroup of G'. In particular, $Imf = f(G)$ is an MΓ-subgroup of G'.*

Proof. It is clear that $f(H)$ is a subgroup of G' for any MΓ-subgroup H of G. Let $x \in M$, $\gamma \in \Gamma$ and $y \in f(H)$. Then $y = f(a)$ for some $a \in H$ and so $x\gamma y = x\gamma f(a) = f(x\gamma a) \in f(H)$, since $x\gamma a \in H$. Thus $f(H)$ is an MΓ-subgroup of G'.

THEOREM 3. *Let $f : G \to G'$ be an MΓ-homomorphism of MΓ-groups and let $K = Kerf$. For any MΓ-subgroup H' of G', $f^{-1}(H') = \{x \in G | f(x) \in H'\}$ is an MΓ-subgroup of G containing K.*

Proof. Note that $f^{-1}(H')$ is a subgroup of G. Let $x \in M$, $\gamma \in \Gamma$ and $a \in f^{-1}(H')$. Then $f(x\gamma a) = x\gamma f(a) \in H'$ because H' is an MΓ-subgroup of G' and $f(a) \in H'$. It follows that $x\gamma a \in f^{-1}(H')$, so that $f^{-1}(H')$ is an MΓ-subgroup of G. Clearly K is contained in $f^{-1}(H')$, completing the proof.

THEOREM 4. *Let $f : G \to G'$ be an MΓ-monomorphism of MΓ-groups. For any MΓ-subgroup H of G, H is MΓ-isomorphic to $f(H)$. In particular, G is MΓ-isomorphic to $Im(f)$ which is equal to $f(G)$.*

Proof. By Theorem 2, $f(H)$ is an MΓ-subgroup of G'. Now we define a map $\phi : H \to f(H)$ by $\phi(h) = f(h)$. Then ϕ is an MΓ-monomorphism from H onto $f(H)$. So we have that H is MΓ-isomorphic to $f(H)$.

THEOREM 5. *Let G and G' be MΓ-groups and let $f : G \to G'$ be an MΓ- homomorphism. Then $Kerf$ is an MΓ-ideal of G.*

Proof. Clearly $(Kerf, +)$ is a normal divisor of $(G, +)$. Let $x \in M$, $\gamma \in \Gamma$, $h \in Kerf$ and $g \in G$. Then

$$\begin{aligned} f(x\gamma(g+h) - x\gamma g) &= f(x\gamma(g+h)) - f(x\gamma g) \\ &= x\gamma f(g+h) - x\gamma f(g) \\ &= x\gamma(f(g) + f(h)) - x\gamma f(g) \\ &= x\gamma f(g) - x\gamma f(g) \\ &= 0. \end{aligned}$$

and so $x\gamma(g+h) - x\gamma g \in Kerf$. Thus $Kerf$ is an MΓ-ideal of G. This completes the proof.

Now we construct the factor MΓ-group. Let H be an MΓ-ideal of an MΓ-group G. Then G/H is a factor group under the operation

$$(g_1 + H) + (g_2 + H) = (g_1 + g_2) + H$$

for any $g_1 + H$, $g_2 + H \in G/H$. For all $x, y \in M$, $\gamma, \mu \in \Gamma$ and $g + H \in G/H$, we define $x\gamma(g + H) = x\gamma g + H$. Then we have

$$\begin{aligned} (x+y)\gamma(g+H) &= (x+y)\gamma g + H \\ &= (x\gamma g + y\gamma g) + H \\ &= (x\gamma g + H) + (y\gamma g + H) \\ &= x\gamma(g+H) + y\gamma(g+H). \end{aligned}$$

and

$$\begin{aligned} x\gamma(y\mu(g+H)) &= x\gamma(y\mu g + H) \\ &= x\gamma(y\mu g) + H \\ &= (x\gamma y)\mu g + H \\ &= (x\gamma y)\mu(g+H). \end{aligned}$$

Thus G/H is an MΓ-group. Hence we get the following theorem.

THEOREM 6. *Let G be an $M\Gamma$-group and let H be an $M\Gamma$-ideal of G. Then the cosets of H form an $M\Gamma$-group G/H whose binary operations are defined by*

$$(a+H)+(b+H)=(a+b)+H$$

and

$$x\gamma(a+H)=x\gamma a+H$$

for all $x \in M, \gamma \in \Gamma$ and $a+H, b+H \in G/H$. This $M\Gamma$-group G/H is called the factor $M\Gamma$-group of G by H.

THEOREM 7. *Let H be an $M\Gamma$-ideal of an $M\Gamma$-group G. Then $f: G \to G/H$ given by $f(g)=g+H$ is an $M\Gamma$-homomorphism with kernel H.*

Proof. Clearly f is a group homomorphism of G into G/H. For all $x \in M$, $\gamma \in \Gamma$ and $g \in G$, $f(x\gamma g)=x\gamma g+H=x\gamma(g+H)=x\gamma f(g)$, so f is an MΓ-homomorphism. Moreover

$$g \in Kerf \Longleftrightarrow f(g)=H \Longleftrightarrow g+H=H \Longleftrightarrow g \in H.$$

Thus $Kerf=H$. This completes the proof.

THEOREM 8. *Let $f: G \to G'$ be an $M\Gamma$-homomorphism with kernel H. Then $f(G)$ is an $M\Gamma$-group, and the map $\phi: G/H \to f(G)$ given by $\phi(g+H)=f(g)$ is an $M\Gamma$-isomorphism. If $\psi: G \to G/H$ is the $M\Gamma$-homomorphism given by $\psi(g)=g+H$, then for each $g \in G$, we have $f(g)=(\phi\psi)(g)$.*

Proof. Clearly, G is an MΓ-subgroup of G. By Theorem 2, $f(G)$ is an MΓ-subgroup of G' and so $f(G)$ is an MΓ-group. If a map ϕ : $G/H \to f(G)$ is defined by $\phi(g+H)=f(g)$ for $g \in G$, then ϕ is a group isomorphism. For any $g+H \in G/H$, $x \in M$ and $\gamma \in \Gamma$,

$$\begin{aligned}\phi(x\gamma(g+H)) &= \phi(x\gamma g+H)\\ &= f(x\gamma g)\\ &= x\gamma f(g)\\ &= x\gamma\phi(g+H).\end{aligned}$$

Thus ϕ is an MΓ-isomorphism. Next, for each $g \in G$,

$$(\phi\psi)(g) = \phi(\psi(g)) = \phi(g + H) = f(g).$$

Thus we have $f = \phi\psi$. This completes the proof.

THEOREM 9. *Let G be an MΓ-group. If S is an MΓ-subgroup of G and H is an MΓ-ideal of G, then*

(1) *the set $S + H = \{s + h | s \in S, h \in H\}$ is an MΓ-subgroup of G and $S \cap H$ is an MΓ-ideal of G, and*

(2) $G/(S \cap H) \simeq (S + H)/H$.

Proof. Clearly $S + H$ is a subgroup of G. For all $x \in M, \gamma \in \Gamma$ and $s + h \in S + H$, $x\gamma(s+h) = x\gamma(s+h) - x\gamma s + x\gamma s \in S + H$, since H is an MΓ-ideal and S is an MΓ-subgroup of G. Thus $S+H$ is an MΓ-subgroup of G. Now consider the MΓ-homomorphism $\pi : G \to G/H, \pi(g) = g+H$ and define a map $f : S \to G/H, f(s) = \pi(s) = s + H$. Then f is an MΓ-homomorphism. Also

$$Ker f = \{s \in S | f(s) = H\} = \{s \in S | s + H = H\} = S \cap H$$

and

$$Im f = \{(s + h) + H \in G/H | s + h \in S + H\} = (S + H)/H.$$

By Theorem 5 and Theorem8, $S \cap H$ is an MΓ–ideal of G and

$$\phi : G/(S \cap H) \to (S + H)/H, \pi(g + (S \cap H)) = g + H$$

is an MΓ-isomorphism. So we have $S/(S \cap H) \simeq (S + H)/H$. This completes the proof.

THEOREM 10. *Let G be an MΓ-group and let H, K be MΓ-ideals of G. If $H \subseteq K$, then $K/H = \{a + H | a \in K\}$ is an MΓ-ideal of G/H and $(G/H)/(K/H) \simeq G/K$.*

Proof. Clearly K/H is a normal divisor of G/H. For all $x \in M, \gamma \in \Gamma$, $a + H \in K/H$ and $g + H \in G/H$,

$$\begin{aligned} x\gamma((g + H) + (a + H)) - x\gamma(g + H) &= x\gamma((g + a) + H) - x\gamma(g + H) \\ &= (x\gamma(g + a) + H) - (x\gamma g + H) \\ &= x\gamma(g + a) - x\gamma g) + H \in K/H. \end{aligned}$$

Thus K/H is an MΓ-ideal of G/H. Next, define a map

$$f : G/H \to G/K, f(g+H) = g+K.$$

Then the map f is well defined. In fact, let $g_1 + H = g_2 + H$ for any $g_1, g_2 \in G$. Then we have $g_1 - g_2 \in H \subseteq K$ and so $g_1 + K = g_2 + K$. Thus $f(g_1 + H) = f(g_2 + H)$. For any $g_1 + H, g_2 + H \in G/H$,

$$\begin{aligned} f((g_1 + H) + (g_2 + H)) &= f((g_1 + g_2) + H) \\ &= (g_1 + g_2) + K \\ &= (g_1 + K) + (g_2 + K) \\ &= f(g_1 + H) + f(g_2 + H). \end{aligned}$$

and

$$\begin{aligned} f(x\gamma(g_1 + H)) &= f(x\gamma g_1 + H) \\ &= x\gamma g_1 + K \\ &= x\gamma(g_1 + K) \\ &= x\gamma f(g_1 + H). \end{aligned}$$

Thus f is an MΓ-homomorphism. Moreover,

$$\begin{aligned} Ker f &= \{g + H | f(g + H) = 0 + K\} \\ &= \{g + H | g + K = K\} \\ &= \{g + H | g \in K\} \\ &= K/H. \end{aligned}$$

and

$$\begin{aligned} Im f &= \{f(g + H) | g + H \in G/H\} \\ &= \{g + K | g \in G\} \\ &= G/K. \end{aligned}$$

Thus, by Theorem 8, we have $(G/H)/(K/H) \simeq Im f = G/K$. This completes the proof.

THEOREM 11. *Let $f : G \to G'$ be an onto MΓ-homomorphism and let $K = Kerf$. Then we have the following:*

(1) *For any MΓ-subgroup H of G, $f(H)$ is an MΓ-subgroup of G'. Moreover,if H is an MΓ-ideal of G, then $f(H)$ is an MΓ-ideal of G'.*
(2) *For any MΓ-subgroup H' of G', $f^{-1}(H') = \{x \in G | f(x) \in H'\}$ is an MΓ-subgroup and $K \subseteq f^{-1}(H')$. Also, if H' is an MΓ-ideal of G', then $f^{-1}(H')$ is an MΓ-ideal of G.*
(3) *For any MΓ-subgroup H of G, $f^{-1}(f(H)) = H + K$ and for any MΓ-subgroup H' of G', $f(f^{-1}(H')) = H'$.*
(4) *Let X be the set of all MΓ-subgroups(MΓ-ideals) of G containing K and let Y be the set of all MΓ-subgroups(MΓ-ideals) of G'. Then there is one-to-one mapping of X onto Y.*

Proof.

(1) By Theorem 2, if H is an MΓ-subgroup of G, then $f(H)$ is an MΓ-subgroup of G'. Let H be an MΓ-ideal of G. For all $x \in M, \gamma \in \Gamma$, $f(a) \in f(H)(a \in H)$ and $g' \in G'$,

$$\begin{aligned} x\gamma(g' + f(a)) - x\gamma g' &= x\gamma(f(g) + f(a)) - x\gamma f(g) \\ &= x\gamma f(g + a) - f(x\gamma g) \\ &= f(x\gamma(g + a) - x\gamma g) \in f(H) \end{aligned}$$

for some $g \in G$, since H is an MΓ-ideal of G and $a \in H$. Thus $f(H)$ is an MΓ-ideal of G'.

(2) By Theorem 3, if H' is an MΓ-subgroup of G', then $f^{-1}(H')$ is an MΓ-subgroup of G and $K \subseteq f^{-1}(H')$. Let H' be an MΓ-ideal of G'. Clearly $f^{-1}(H')$ is a normal divisor of G. For all $x \in M, \gamma \in \Gamma, a \in f^{-1}(H')$ and $g \in G$,

$$\begin{aligned} f(x\gamma(g + a) - x\gamma g) &= f(x\gamma(g + a)) - f(x\gamma g) \\ &= x\gamma f(g + a) - x\gamma f(g) \\ &= x\gamma(f(g) + f(a)) - x\gamma f(g) \in H', \end{aligned}$$

since H' is an MΓ-ideal of G' and $f(a) \in H'$.

(3) If H is an MΓ-subgroup of G, by Theorem 2 and 3 $f^{-1}(f(H))$ is an MΓ-subgroup of G. If $a \in f^{-1}(f(H))$, then $f(a) \in f(H)$ and so $f(a) = f(h)$ for some $h \in H$. So we have $a - h \in Ker f = K$. Then, since $a \in a + K = h + K \subseteq H + K, f^{-1}(f(H)) \subseteq H + K$. Conversely, for any $h \in H$ and $k \in K$, since $f(h + k) = f(h) + f(k) = f(h) + 0 = f(h) \in f(H), h + k \in f^{-1}(f(H))$ and so $H + K \subseteq f^{-1}(f(H))$. Therefore $f^{-1}(f(H)) = H + K$. Next, if H' is an MΓ-subgroup of G', then since f is onto mapping, $f(f^{-1}(H')) = H' \cap f(G) = H' \cap G' = H'$.

(4) By (3), if H is an MΓ-subgroup of G containing K, $f^{-1}(f(H)) = H + K = H$. In particular, if H is an MΓ-ideal of G, then $f(H)$ is an MΓ-ideal of G'. Also if H' is an MΓ-subgroup of G', $f(f^{-1}(H')) = H'$ and $f(H')$ is an MΓ-ideal of G for an MΓ-ideal H' of G'. This completes the proof.

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