Pusan Kyŏngnam Math. J. **12**(1996), No. 1, pp. 41–45

SOME PROPERTIES ON FAITHFUL R–GROUPS

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1. Introduction

Let R be a (left) near-ring (see G. Pilz [5]) let $(G, +)$ be a group. Define

$$M(G) := \{f : G \to G\}$$

be the set of all maps from G to G, with addition defined pointwise:

$$x(f+g) := xf + xf$$

for all x in G and multiplication as the usual composition of maps

$$x(fg) := (xf)g.$$

Then $M(G)$ becomes a (left) near-ring which is called the near-ring of all mappings on a group G.

In this paper, many of the groups that will occur will be written additively. This is not to be taken to imply commotivity. Indeed most of the groups which we will concern with will be noncommutative. Again let $(G, .)$ be a group. Define

$$M_0(G) := \{f \in M(G); 0f = 0\}$$

be the set of all maps from G to G which map the identity of G to itself. We see that $M_0(G)$ is a subnear- ring of $M(G)$, which is known as the near-ring of all zero preserving mappings on G with addition and multiplication are defined as in $M(G)$.

If R is a ring and $R[x]$ is the set of all polynomials in one indeterminate over R. Define addition in $R[x]$ in the usual way, and define

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Received Feburary 9,1996.

This paper was supported (in part) by NON DIRECTED RESEARCH FUND, Korea Research Foundation .

composition " $\circ$ " by $f \circ g := f(g)$, where$f, g \in R[x]$. Then $R[x]$ becomes a right near-ring which is called a right near-ring of polynomials over a ring R, (see Lausch and Nobauer [2]). We note that $R[x]$ is a near-ring whose additive group is commutative. There is also the same case for $M(G)$ and $M_0(G)$ if G is abelian. This prompts the following consepts:

A near-ring $(R, +, \cdot)$ is called abelian if $(R, +)$ is an abelian group, commutative if $(R, \cdot)$ is a commutative semigroup, zero-symmetric near-ring if $0a = 0$ for all $a \in R$ and constant near-ring if $ab = b$ for all a, b in R. Clearly $M_0(G)$ is a zero-symmetric near-ring, and constant subnear-ring of R is the set of all elements:

$$\begin{aligned}\{0a; a \in R\} &= \{b \in R \ ; ab = b \text{ for all } a \in R\} \\ &= \{b \in R \ ; b = 0c \text{ for some } c \in R\}\end{aligned}$$

For a given group $(G, +)$, define a multiplication $\circ$ on G by $x \circ y = y$ for all x, y in G. With this multiplition, $(G, +, \circ)$ is a constant near-ring on G. Define a second multiplition $*$ on G by $x * y = 0$ if $x = 0$, $= y$ otherwise. With this second multiplition, $(G, +, *)$ is a zero-symmetric near-ring on G.

We have already known that every near-ring can be considered as a subnear-ring of a near-ring $M(G)$ of all mappings on a group G. A group G is called a (right) R-group if there is a near-ring homomorphism

$$\theta : (R, +, \cdot) \to (M(G), +, \cdot).$$

Such a homomorphism θ is called a representation of R. In R-group theory,there is one important and almost universally used convention. If G is an R-group,write xr for $x(r\theta)$ for all $x \in G, r \in R$.

2.Properties on Faithful R-Groups

Let G be an R-group and K, K_1 and K_2 subsets of G. Define

$$(K_1 : K_2) := \{a \in R; K_2 a \subset K_1).$$

We abbreviate that for $x \in G$

$$(\{x\} : K_2) =: (x : K_2)$$

Similarly for $(K_1 : x)$. $(0 : K)$ is called the annihilator of K ,denoted by $A(K)$. We say that G is a faithful R-group or that R acts faithfully on G if $A(G) = 0$, taht is, $(0 : G) = 0$. A subgroup H of G such that $xa \in H$ for all $x \in H, a \in R$, is called an R-subgroup of G. An R-ideal(simply, ideal) of G is a normal subgroup N of G such that

$$(g + x)a - ga \in N$$

for all $g \in G, x \in N, a \in R$. If R has an identy 1_R, then we say that G is a unital R-group if $g1_R = g$ for all $g \in G$. (see J.D.P.Meldrum [3]).

LEMMA 2.1. *Let G be an R-group and K_1 and K_2 be subsets of G. Then we have the following conditions:*

(1) *If K_1 is a normal subgroup of G, then $(K_1 : K_2)$ is a normal subgroup of a near-ring R.*
(2) *If K_1 is an R-subgroup of G, then $(K_2 : K_2)$ is an R-subgroup of R as an R-group.*
(3) *If K_1 is an ideal of G and K_2 is an R-subgroup of G, then $(K_1 : K_2)$ is an ideal of R.*

Proof. (1) and (2) are easily proved (see J.D.P.Meldrum [4]). We will prove only (3) : Using the condition (1),$(K_1 : K_2)$ is a normal subgroup of R. Let $a \in (K_1 : K_2)$ and $r \in R$ then

$$K_2(ra) = (K_2r)a \subset K_2a \subset K_1,$$

so that $ra \in (K_1 : K_2)$. Whence $(K_1 : K_2)$ is a left ideal of R.

Next, let $r_1, r_2 \in R$ and $a \in (K_1 : K_2)$. then

$$k\{(a + r_1)r_2 - r_1r_2\} = (ka + kr_1)r_2 - kr_1r_2 \in K_1$$

for all $k \in K_2$, since $K_2a \subset K_1$ and K_1 is an ideal of G. Thus $(K_1 : K_2)$ is a right ideal of R. Therefore $K_1 : K_2)$ is a (two-sided) ideal of R. □

COROLLARY 2.2([5]). *Let R be a near-ring and G an R-group.*

(1) *For any $x \in G, (0 : x)$ is a right ideal of R.*
(2) *For any R-subgroup K of G, $(0 : K)$ is an ideal of R.*
(3) *For any subset K of G, $(0 : K) = \bigcap_{x \in K}(0 : x)$.*

PROPOSITION 2.3. *Let R be a near-ring and G be an R-group. Then we have the following conditions:*

(1) *$A(G)$ is a two-sided ideal of R. Moreover G is a faithful $R/A(G)$-group.*

(2) *For any $x \in G$, we get $xR \cong R/(0:x)$ as R-groups.*

Proof. (1) By corollary 2.2 and lemma 2.1, $A(G)$ is a two-sided ideal of R.

We now make G an $R/A(G)$-group by defining, for $x \in R, r+A(G) \in R/A(G)$, the action $x(r + A(G)) = xr$. If $r + A(G) = r' + A(G)$, then $-r' + r \in A(G)$ hence $x(-r' + r) = 0$ for all x in G, that is to say, $xr = xr'$. This tells us that

$$x(r + A(G)) = xr = xr' = x(r' + A(G));$$

thus the action of $R/A(G)$ on G has been shown to be well defined. The verification that this defiines the structure of an $R/A(G)$-group on$G$ is a routine triviality, we leave to the reader. Finally, to see that G is a faithful $R/A(G)$-group. We note that if $x(r + A(G)) = 0$ for all $x \in G$. then by the definition of $R/A(G)$-group structure, we have $xr = 0$.

Hence $r \in A(G)$. This says that only the zero element of $R/A(G)$ annihilates all of G. Thus G is a faithful $R/A(G)$-group.

(2) For any $x \in G$, clearly xR is an R-subgroup of G. The map $\phi : R \longrightarrow xR$ defined by $\phi(r) = xr$ is an R-ephimorphim, so that from the isomorphism theorem and the kernel of ϕ is $(0:x)$, we deduce that $xR \cong R/(0:x)$ as R-groups. $\square$

PROPOSITION 2.4. *Let R be a near-ring and G an R-group. Then $R/A(G)$ is near-ring isomorphic to a subnear-ring of $M(G)$.*

Proof. For any $a \in R$, we define $T_a : G \longrightarrow G$ by $xT_a = xa$ for each $x \in G$. Then T_a is a mapping from G to G, that is, T_a is in $M(G)$. Consider the mapping $\phi : R \longrightarrow M(G)$ defined by $\phi(a) = T_a$. Going back to the definition of an R-group, we see that

$$\phi(a + b) = \phi(a) + \phi(b) \text{ and } \phi(ab) = \phi(a)\phi(b),$$

that is to say, ϕ is a near-ring homomorphism of R into $M(G)$.

Finally we must to show that $Ker\phi = A(G)$: Indeed if $a \in A(G)$, then $Ga = 0$ hence $0 = T_a = \phi(a)$, namely, $a \in Ker\phi$. On the other hand if $a \in Ker\phi$, then $T_a = 0$ leading to $Ga = GT_a = 0$, that is, $a \in A(G)$. Therefore the image of R in $M(G)$ is a near-ring isomorphic to $R/A(G)$, by the first isomorphism theorem on R-groups. Our proof is complete. □

Now we have very important following statement as the corresponding results from ring theory.

COROLLARY 2.5. *If G is a faithful R-group, where R is any near-ring, then R is embedded in $M(G)$.*

Proof. In the proposition 2.4, we see that $A(G) = 0$, since G is faithful. □

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