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EVALUATION OF $\int_0^{\frac{\pi}{4}} \log \sin t \, dt$

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The following definite integral formula is well known:

$$\int_0^{\frac{\pi}{2}} \log \sin t \, dt = -\frac{\pi}{2} \log 2,$$

which appears in any reasonable elementary calculus book and mathematical formula books (see [5, p. 99. Entry 15.102]). However we cannot locate the seat in which the definite integral

$$\int_0^{\frac{\pi}{4}} \log \sin t \, dt$$

was evaluated. We find it difficult to evaluate this integral in usual and elementary ways. So we tried to evaluate this integral using the theory of special functions. Note that every logarithm log means the natural logarithm.

The double Gamma function had been defined and studied by Barnes [1] and others in about 1900, not appearing in the tables of the most well-known special functions, cited in the exercise by Whittaker and Watson [6, p. 264]. Recently this function has been revived according to the study of determinants of Laplacians (see [2]).

Barnes defined the double Gamma function Γ_2:

$$\{\Gamma_2(z+1)\}^{-1} = G(z+1) = (2\pi)^{\frac{z}{2}} e^{-\frac{1}{2}\{(1+\gamma)z^2+z\}} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right)^k e^{-z+\frac{z^2}{2k}},$$

where γ is the Euler-Mascheroni's constant defined by

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) \cong 0.577\,215\,664\ldots. \tag{1}$$

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The double Gamma function satisfies $G(1) = 1$ and $G(z+1) = \Gamma(z)G(z)$ for every complex z where Γ is the well-known Gamma function:

$$(2) \qquad \Gamma(z+1)^{-1} = e^{\gamma z} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-\frac{z}{k}}.$$

Stirling's expansion for $z \to +\infty$ is given [1]:

$$\log G(1+z) = z^2 \left(\frac{\log z}{2} - \frac{3}{4}\right) + \frac{z}{2}\log(2\pi) - \frac{\log z}{12} + \frac{1}{12} - \log A + O(1/z),$$

where A is called Glaisher's (or Kinkelin's) constant defined by

$$(3) \quad \log A = \lim_{n\to\infty} \left\{ \log(1^1 2^2 \cdots n^n) - (\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}) \log n + \frac{n^2}{4} \right\},$$

the numerical value of A being $1.282427130\cdots$ and Glaisher studied extensively the constant A in his several papers (see [4]).

The Maclaurin summation formula [3, p. 117] is given here for easy reference:

$$(4) \ \sum u_x = C + \int u_x dx - \frac{1}{2}u_x + \frac{B_1}{2!}\frac{d}{dx}u_x - \frac{B_3}{4!}\frac{d^3}{dx^3}u_x + \frac{B_5}{6!}\frac{d^5}{dx^5}u_x - \cdots$$

where C is an arbitrary constant to be determined in each special case, $\sum u_x = u_{x-1} + u_{x-2} + \cdots + u_a$, u_a is some fixed term of the series and B_{2n-1}, $n = 1, 2, \ldots$ are Bernoulli numbers.

Setting $u_x = (x + 1/4)\log(x + 1/4)$ in (4), and adding u_x to both sides of the resulting equation, and replacing x by n in the last resulting equation yields a mathematical constant C_1:

$$(5) \quad \begin{aligned} \log C_1 = \lim_{n\to\infty} &\left\{ \sum_{k=1}^{n} \left(k+\frac{1}{4}\right) \log\left(k+\frac{1}{4}\right) \right. \\ &\left. - \left(\frac{n^2}{2} + \frac{3n}{4} + \frac{23}{96}\right) \log\left(n+\frac{1}{4}\right) + \frac{n^2}{4} + \frac{n}{8} \right\}, \end{aligned}$$

the numerical value of C_1 being 1.3781.... Similarly setting $u_x = (x - 1/4)\log(x - 1/4)$ in (4) yields another mathematical constant C_2:

$$\begin{aligned}\log C_2 = \lim_{n\to\infty} \Bigg\{ & \sum_{k=1}^{n} \left(k - \frac{1}{4}\right) \log\left(k - \frac{1}{4}\right) \\ & - \left(\frac{n^2}{2} + \frac{n}{4} - \frac{1}{96}\right) \log\left(n - \frac{1}{4}\right) + \frac{n^2}{4} - \frac{n}{8} \Bigg\},\end{aligned} \tag{6}$$

the numerical value of C_2 being 1.1274.... Note that the constant A can also be obtained by setting $u_x = x \log x$ in (4).

Recall the Stirling's formula [3, p. 68]:

$$\frac{1}{2}\log(2\pi) = \lim_{n\to\infty} \left\{ \log n! + n - \left(n + \frac{1}{2}\right) \log n \right\}. \tag{7}$$

Now we will evaluate $-\int_0^{1/4} \log \Gamma(1 + t)dt$ directly using the definition of Gamma function (2) and denote this integral by I: Taking logarithms on both sides of (2) and integrating the resulting equation from 0 to 1/4 yields

$$\begin{aligned} I &= \frac{\gamma}{32} + \sum_{k=1}^{\infty} \int_0^{\frac{1}{4}} \left\{ \log\left(1 + \frac{t}{k}\right) - \frac{t}{k} \right\} dt \\ &= \frac{\gamma}{32} + \lim_{n\to\infty} \sum_{k=1}^{n} \left\{ \left(k + \frac{1}{4}\right) \log\left(k + \frac{1}{4}\right) - \frac{1}{4} - k\log k - \frac{1}{4}\log k - \frac{1}{32k} \right\} \\ &= \frac{\gamma}{32} + \lim_{n\to\infty} \Bigg\{ \sum_{k=1}^{n} \left(k + \frac{1}{4}\right) \log\left(k + \frac{1}{4}\right) - \frac{1}{4}n \\ &\quad - \sum_{k=1}^{n} k\log k - \frac{1}{4}\sum_{k=1}^{n} \log k - \frac{1}{32}\sum_{k=1}^{n} \frac{1}{k} \Bigg\} \\ &= \log C_1 - \log A - \frac{1}{8}\log(2\pi) + \lim_{n\to\infty} \Bigg[\left(\frac{n^2}{2} + \frac{3n}{4} + \frac{23}{96}\right) \log\left(n + \frac{1}{4}\right) \\ &\quad - \left(\frac{n^2}{2} + \frac{3n}{4} + \frac{23}{96}\right) \log n - \frac{n}{8} \Bigg] \end{aligned}$$

$$= \log C_1 - \log A - \frac{1}{8}\log(2\pi)$$
$$+ \lim_{n\to\infty}\left[\left(\frac{n^2}{2} + \frac{3n}{4} + \frac{23}{96}\right)\log\left(1 + \frac{1}{4n}\right) - \frac{n}{8}\right]$$
$$= \log C_1 - \log A - \frac{1}{8}\log(2\pi) + \lim_{n\to\infty}\left[\left\{\frac{n}{8} + \frac{11}{64} + O\left(\frac{1}{n}\right)\right\} - \frac{n}{8}\right]$$
$$= \log C_1 - \log A - \frac{1}{8}\log(2\pi) + \frac{11}{64},$$

where we use (1), (3), (5) and (7) for the fourth equality, and the Maclaurin series of $\log(1+x)$ for the sixth one.

We therefore have

$$\int_0^{\frac{1}{4}} \log\Gamma(1+t)\,dt = \log A - \log C_1 + \frac{1}{8}\log(2\pi) - \frac{11}{64}. \tag{8}$$

Similarly we have

$$\int_0^{\frac{1}{4}} \log\Gamma(1-t)\,dt = \log C_2 - \log A + \frac{1}{8}\log(2\pi) - \frac{5}{64}. \tag{9}$$

Recalling the well-known relation (see [6, p. 239]):

$$\Gamma(1+t)\Gamma(1-t) = \frac{\pi t}{\sin(\pi t)},$$

we obtain

(10)
$$\int_0^{\frac{1}{4}} \log\Gamma(1+t)\,dt + \int_0^{\frac{1}{4}} \log\Gamma(1-t)\,dt = \frac{1}{4}\log\left(\frac{\pi}{4}\right) - \frac{1}{4} - \frac{1}{\pi}\int_0^{\frac{\pi}{4}} \log\sin t\,dt.$$

Finally combining (8) and (9) with (10) yields our desired result

$$\int_0^{\frac{\pi}{4}} \log\sin t\,dt = \pi\left(\log C_1 - \log C_2 - \frac{3}{4}\log 2\right). \tag{11}$$

References

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