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ON A WEIGHTED MAXIMAL MEANS OFF THE LINE $\frac{1}{p} = \frac{1}{q}$

JUNG S. RHEE

One of the main topics in the harmonic analysis is the local smoothig estimates for a certain maximal means. Elias M. Stein introduced the spherical maximal means in his paper [5]. Also J. Bourgain showed that there is a local smoothing estimates for Stein's maximal means in [1]. In recent years, Mockenhaupt, Seeger and Sogge reinforced the local smoothing estimates of Bourgain's circular maximal means(see [2]). In [3], Oberlin studied Stein's maximal estimates off the dual line $\frac{1}{p} = \frac{1}{q}$. A partial soluton of the maximal means off the line $\frac{1}{p} = \frac{1}{q}$ was given by Oberlin in [3].

In this note, we will give a complete solution what Oberlin has expected(see [3]). Actually we give a sharp estimates of a (spherical) maximal means off the dual line for $n \geq 3$ and $\alpha \geq 0$.

Let us define spherical means of (complex) order α by

$$M_t^\alpha f(x) = \int_{R^n} (1 - |y|^2)_+^{\alpha-1} f(x - ty)dy, f \in C_0^\infty(R^n), t > 0.$$

These means are defined for $Re\alpha > 0$, but the definion can be extened to the region $Re\alpha \leq 0$ by analytic continuation. For $Re\alpha \leq 0$, we put $M_t^\alpha f(x) = m_{\alpha,t} * f(x)$ given by $\widehat{M_t^\alpha f}(x) = \widehat{m_\alpha}(tx)\hat{f}(x)$ where $\widehat{m_\alpha}(x) = \pi^{-\alpha+1}|x|^{-\frac{n}{2}-\alpha+1} J_{\frac{n}{2}+\alpha-1}(2\pi|x|)$, J_α is a Bessel function of order α and $m_{\alpha,t}(x) = m_\alpha(\frac{x}{t})t^{-n}, t > 0, x \in R^n$.

Now we consider a maximal function(introduced by Oberlin-see [3]):

$$\mathcal{T}_{p,q}^\alpha f(x) = \sup_{r>0} r^{\frac{n}{p}-\frac{n}{q}} |M_t^\alpha f(x)|, 1 \leq p \leq q \leq \infty.$$

For $n = 2$, it is known that $\{\int_{R^2} (\sup_{r>0} |M_t^\alpha f(x)|)^4 dx\}^{\frac{1}{4}} \leq c\|f\|_4$ if $\alpha > -\frac{1}{8}$ (see [2]). Thus we can have some estimates of operators $\mathcal{T}_{p,q}^\alpha$ by interpolating the above result and (b) in the following theorem.

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THEOREM 1. *(a) Let $\alpha \geq 0$ and $n \geq 3$. Then*

$$\|\mathcal{T}^{\alpha}_{p,q} f\|_{L^q(dx)} \leq c_\alpha \|f\|_{L^p(R^n)}$$

if $(\frac{1}{p}, \frac{1}{q})$ is in the region: $\frac{1}{q} > \frac{1}{n}(\frac{1}{p} - \alpha)$, $\frac{1}{q} > \frac{1}{n-1}(\frac{n+1}{p} - (n-1+2\alpha))$ and $\frac{1}{p} < \frac{n-1}{n} + \frac{1}{n}\alpha$.

(b) For $n = 2$, the above result is true when $\alpha > \frac{1}{6}$.

COMMENT:. *(a) and (b) are sharp in the sense of a necessary condition for $n \geq 2$. For $\alpha = 0$, see a necessary condition in* [3]. *For $\alpha > 0$, we can easily get the above condition using a transform in lemma 2.*

The proof is based on a (Stein's) trsnsform and interpolation theorem.

LEMMA 2. *Let $1 < p \leq q < \infty$.*
Set $\alpha > \alpha' + \frac{1}{q}, \alpha' > -\frac{n}{2} + \frac{1}{2}$ and $\alpha' > \frac{n}{2}(\frac{1}{p} - \frac{1}{q} - 1) + \frac{1}{2q}$. Then

$$t^{\frac{n}{p}-\frac{n}{q}} |M^{\alpha}_t f(x)| \leq C(\alpha, \alpha', p, q)\Big\{\frac{1}{t}\int_0^t |s^{\frac{n}{p}-\frac{n}{q}} M^{\alpha'}_s f(x)|^q ds\Big\}^{1/q}$$

where C depends only on p, q, α and α'.

Proof. By a (Stein's) transform (see [S1]), we have a following calculation:

$$M^{\alpha}_t f(x) = \frac{2}{\Gamma(\alpha - \alpha')}\int_0^1 M^{\alpha'}_{st} f(x)(1-s^2)^{\alpha-\alpha'-1} s^{n+2\alpha'-1} ds$$

for $\alpha > \alpha' > -\frac{n}{2} + \frac{1}{2}$. Then

$$\begin{aligned} & t^{\frac{n}{p}-\frac{n}{q}} |M^{\alpha}_t f(x)| \\ & \leq \frac{2}{\Gamma(\alpha-\alpha')}\int_0^1 (st)^{\frac{n}{p}-\frac{n}{q}} M^{\alpha'}_{st} f(x) s^{n(1-\frac{1}{p}+\frac{1}{q})+2\alpha'-1}(1-s^2)^{\alpha-\alpha'-1} ds \\ & \leq C(\alpha, \alpha', p, q)\Big\{\frac{1}{t}\int_0^t u^{\frac{n}{p}-\frac{n}{q}} |M^{\alpha'}_u f(x)|^q du\Big\}^{\frac{1}{q}} \end{aligned}$$

by Hőlder's inequality. Here,

$$\begin{aligned} & C(\alpha, \alpha', p, q) \\ & = \Big\{\frac{\Gamma(q'(\alpha-\alpha'-1)+1)\Gamma(\frac{q'n}{2}(1-\frac{1}{p}+\frac{1}{q}) + q'\alpha' - \frac{1}{2}(q'-1))}{2\Gamma(q'(\alpha-\alpha'-1)+1+\frac{q'n}{2}(1-\frac{1}{p}+\frac{1}{q}) + q'\alpha' - \frac{1}{2}(q'-1))}\Big\}^{\frac{1}{q'}}. \end{aligned}$$

LEMMA 3. *Let* $\alpha' = -n+1+\frac{n}{p}$ *and* $\frac{1}{2}\le\frac{1}{p}\le 1$. *Then*

$$\left\{\int_{R^n}\int_0^\infty (r^{\frac{n}{p}-\frac{n}{q}}|M_r^{\alpha'}f(x)|)^q\frac{dr}{r}dx\right\}^{\frac{1}{q}}\le C\|f\|_p$$

for $\frac{1}{q}=\frac{n-1}{n+1}(1-\frac{1}{p})$.

Proof. Consider a family of operators $S_z : L^p(R^n, dx)\longrightarrow L^q(R_+^{n+1}, \frac{dr}{r}dx)$ given by $S_zf(x,r)=r^{\frac{n(z+1)}{(n+1)}}M_r^{\frac{z}{2}+1-\frac{n}{2}}f(x)$ for any complex number z.
Then we have the following known results by TT^*-method (see Lemma 5 in [4]):

$$\begin{aligned}\|S_{it}f(x,r)\|_2 &\le \left\{\int_{R^n}\int_0^\infty (r^{\frac{n}{(n+1)}}|M_r^{\frac{it}{2}-\frac{n}{2}+1}f(x)|)^{\frac{2(n+1)}{(n-1)}}\frac{dr}{r}dx\right\}^{\frac{(n-1)}{2(n+1)}}\\ &\le ca^{\frac{|t|}{2}}\|f\|_2\end{aligned}$$

for some constants c and a .

Since $|S_{n+it}f(x,r)|=r^{n+\frac{itn}{n+1}}M_r^{1+\frac{it}{2}}f(x)$, we can easily show that

$$\|S_{n+it}f\|_{L^\infty(R_+^{n+1},\frac{dr}{r}dx)}\le ce^{|t|}\|f\|_{L^1(R^n)}.$$

Hence the complex interpolation theorem gives our results. Now we are ready to finish Theorem 1.

Proof of theorem 1. Let $n\ge 2$. By the complex interpolation theorem between $(0,0)$ and $(1,\infty)$, we can see easily that
$\|\sup_{r>0} r^{\frac{n}{p}}|M_r^\sigma f(x)|\|_\infty\le c\|f\|_p$ if $\frac{1}{p}<\sigma$ and $0<\sigma<1$.

It is known that $\|\sup_{r>0}|M_r^\sigma f(x)|\|_p\le c\|f\|_p$ if $\frac{1}{p}<\frac{n-1}{n}+\frac{\sigma}{n}$ and $0<\sigma<1$ by Stein's maximal theorem(see [5]).

In order to interpolate points in the line $\frac{1}{p}+\frac{1}{q}=1$ we will use Oberlin's method-see [3] for $n\ge 3$.

Consider a family of operators $T_z : L^p(R^n)\longrightarrow L^q(dx, L^s(\frac{dr}{r}))$ given by $T_zf(x,r)=r^zM_r^{\alpha(z)}f(x)$, where $L^q(dx,L^s(\frac{dr}{r}))$ is a mixed normed space with norm

$\|g\|_{q,s} = \{\int_{R^n}(\int_0^\infty |g(x,r)|^s \frac{dr}{r})^{\frac{q}{s}} dx\}^{\frac{1}{q}}$. Let p be fixed with $\frac{1}{2} \le \frac{1}{p} < \frac{n-1}{n} + \frac{\sigma}{n}$. Put $\alpha(z) = 1 + \frac{(1-\sigma)p}{2(p-1)}(\frac{z}{n} - 1)$. Choose $\epsilon > 0$ such that $\epsilon = \frac{1}{2}(n - \frac{(1-\sigma)p}{p-1})$.

Then we obtain the following estimates by complex interpolation and the choice of ϵ:

$$\|T_{\frac{n}{p}-\frac{n}{p'}} f\|_{p',\infty} = \{\int_{R^n} (\sup_{r>0} r^{\frac{n}{p}-\frac{n}{p'}} |M_r^\sigma f(x)|)^{p'} dx\}^{\frac{1}{p'}} \le c\|f\|_p$$

if $\frac{1}{2} \le \frac{1}{p} < \frac{n-1}{n} + \frac{\sigma}{n}$.

We will interpolate points on typediagram in the line $\frac{1}{q} = \frac{n-1}{n+1}(1-\frac{1}{p})$. Set $\frac{-n^2+2n+1}{2(n+1)} \le \alpha \le 1$. Lemma 2 and Lemma 3 gives:

$$\begin{aligned}
&\|\sup_{r>0} r^{\frac{n}{p}-\frac{n}{q}} |M_r^\sigma f(x)|\|_{L^q(R^n)} \\
&\le C\{\int_{R^n} (\sup_{r>0}\{\frac{1}{r}\int_0^r |s^{\frac{n}{p}-\frac{n}{q}} M_s^{\sigma'} f(x)|^q ds\}^{\frac{1}{q}})^q dx\}^{\frac{1}{q}} \\
&\le C\{\int_{R^n}\int_0^\infty |s^{\frac{n}{p}-\frac{n}{q}} M_s^{\sigma'} f(x)|^q \frac{ds}{s} dx\}^{\frac{1}{q}} \\
&\le C\|f\|_{L^p(R^n)}
\end{aligned}$$

if $\sigma > \sigma' + \frac{1}{q} = \alpha$, $\frac{1}{p} = \frac{n^2-n}{n^2+1} + \frac{\alpha(n+1)}{n^2+1}$ and $\frac{1}{q} = \frac{n-1}{n+1}(1 - \frac{1}{p})$.

Thus we have $\|\sup_{r>0} r^{\frac{n}{p}-\frac{n}{q}} |M_r^0 f(x)|\|_q \le c\|f\|_p$ for $\frac{1}{p} < \frac{n^2-n}{n^2+1}$ and $\frac{1}{q} = \frac{n-1}{n+1}(1 - \frac{1}{p})$ since we can choose $\alpha < 0$ such that $\frac{1}{p} = \frac{n^2-n}{n^2+1} + \frac{\alpha(n+1)}{n^2+1} < \frac{n^2-n}{n^2+1}$.

If $\sigma > 0$ we will take $\alpha < \sigma$ such that $\frac{1}{p} = \frac{n^2-n}{n^2+1} + \frac{\alpha(n+1)}{n^2+1} < \frac{n^2-n}{n^2+1} + \frac{\sigma(n+1)}{n^2+1}$. Then we obtain $\|\sup_{r>0} r^{\frac{n}{p}-\frac{n}{q}} |M_r^\sigma f(x)|\|_q \le C_\sigma\|f\|_p$ if $\frac{1}{p} < \frac{n^2-n}{n^2+1} + \frac{\sigma(n+1)}{n^2+1}$, $\frac{1}{q} = \frac{n-1}{n+1}(1 - \frac{1}{p})$.

For $n = 2$, we can get $\{\int_{R^n}\int_0^\infty |r^{\frac{2}{3}} M_r^0 f(x)|^6 \frac{dr}{r} dx\}^{\frac{1}{6}} \le C\|f\|_2$ by Lemma 2. Then we have $\{\int_{R^n} |\sup_{r>0} r^{\frac{2}{3}} M_r^\sigma f(x)|^6 dx\}^{\frac{1}{6}} \le C\|f\|_2$ if $\sigma > \frac{1}{6}$. Thus it is natural to obtain the reults in (b) by the copy of the previous proof.

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DEPARTMENT OF MATHEMATICS
PUSAN UNIVERSITY OF FOREIGN STUDIES,55–1
UAMDONG, NAMKU, PUSAN 608-738, KOREA
e-mail: rhee@taejo.pufs.ac.kr