Honam Mathematical J. 18(1996), No. 1,pp. 33–42

A NOTE ON THE PROJECTIVE REPRESENTATIONS OF A FINITE GROUP

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Let G be a finite group and F a field. The group of nonzero scalar matrices $Z(n, F) \subset GL(n, F)$ is the center of $GL(n, F)$ $(= Z(GL(n, F)))$. The projective general linear group $PGL(n, F)$ is defined by $GL(n, F)/Z(n, F)$. Let

$$\pi : G \longrightarrow PGL(n, F)$$

be a group homomorphism. Then π is a composition of a group homomorphism

$$\kappa : GL(n, F) \longrightarrow PGL(n, F)$$

and a mapping

$$\mathfrak{X} : G \longrightarrow GL(n, F)$$

satisfying that for all $g, h \in G$ and some function $\alpha : G \times G \longrightarrow F$,

$$\mathfrak{X}(g)\mathfrak{X}(h) = \alpha(g, h)\mathfrak{X}(gh). \tag{1}$$

A projective F-representation $\mathfrak{X}$ of G is a function

$$\mathfrak{X} : G \longrightarrow GL(n, F)$$

satisfying (1). Here n is the *degree* of $\mathfrak{X}$ and the function α is the *associated F-factor set* of $\mathfrak{X}$.

Received March 16 1996.

The study of projective representations arises in the study of ordinary representations. For example, consider $N \lhd G$ and an irreducible $\mathbb{C}$-representation $\mathfrak{Y}$ of N such that for all $g, h \in G$, $\mathfrak{Y}^g(h) = \mathfrak{Y}(ghg^{-1}) = \mathfrak{Y}(h)$. Then, there exists a projective $\mathbb{C}$-representation $\mathfrak{X}$ of G satisfying that for all $n \in N$ and $g \in G$,

$$\begin{aligned} &\text{(a)} \quad \mathfrak{X}(n) = \mathfrak{Y}(n) \\ &\text{(b)} \quad \mathfrak{X}(ng) = \mathfrak{X}(n)\mathfrak{X}(g) \\ &\text{(c)} \quad \mathfrak{X}(gn) = \mathfrak{X}(g)\mathfrak{X}(n), \end{aligned}$$

where $\mathbb{C}$ is the field of complexes (cf. [1]).

In this note, we shall describe the relation between the cohomology theory and the associated factor sets of G, and prove some properties of the associated factor sets of G (see Proposition 2, 3, and Theorem 4).

LEMMA 1. *Let $\alpha : G \times G \longrightarrow F^{\times}$ be the associated factor set, of a projective F-representation of G. Then*

(1) *For all $x, y, z \in G$,*

$$\alpha(xy, z)\alpha(x, y) = \alpha(x, yz)\alpha(y, z), \qquad \textit{and}$$

(2) $\alpha(1.1) = \alpha(1, x) = \alpha(x, 1)$, *where* $F^{\times} = F - \{0\}$.

Proof. Let $\mathfrak{X}$ be the projective representation. Note first

$$\begin{aligned} \mathfrak{X}(x)\mathfrak{X}(y)\mathfrak{X}(z) &= \alpha(x, y)\mathfrak{X}(xy)\mathfrak{X}(z) \\ &= \alpha(x, y)\alpha(xy, z)\mathfrak{X}(xyz) \\ &= \mathfrak{X}(x)\alpha(y, z)\mathfrak{X}(yz) \\ &= \alpha(y, z)\alpha(x, yz)\mathfrak{X}(xyz). \end{aligned}$$

This implies that $\alpha(xy, z)\alpha(x, y) = \alpha(x, yz)\alpha(y, z)$. Thus, (1) holds. In addition, from (1) it follows that

$$\alpha(1 \cdot 1, x)\alpha(1, 1) = \alpha(1, 1 \cdot x)\alpha(1, x),$$

and then $\alpha(1, 1) = \alpha(1, x)$ for all $x \in G$. Symmetrically, it also follows that $\alpha(1, 1) = \alpha(x, 1)$. □

For a possibly infinite abelian group A, an *A-factor set* of G is a function $\alpha : G \times G \longrightarrow A$ such that

$$\alpha(xy,z)\alpha(x,y) = \alpha(x,yz)\alpha(y,z),$$

for all $x,y,z \in G$.

Let α be an $F^{\times}$-factor set of G. The twisted group algebra $F^{\alpha}[G]$ with respect to α is defined as follows. Let $F^{\alpha}[G]$ be the F-vector space with basis $\{\bar{g} \mid g \in G\}$. Define multiplication in $F^{\alpha}[G]$ by $\bar{g} \cdot \bar{h} = \overline{gh}\,\alpha(g,h)$ and extend via the distributive law. Then $F^{\alpha}[G]$ is a group algebra.

For all $g,h,z \in G$, we have to prove that

$$(\bar{g} \cdot \bar{h}) \cdot \bar{z} = \bar{g} \cdot (\bar{h} \cdot \bar{z}).$$

But

$$(\bar{g} \cdot \bar{h}) \cdot \bar{z} = \overline{gh}\,\alpha(g,h)\,\bar{z} = \overline{ghz}\,\alpha(g,h)\,\alpha(gh,z),$$
$$\bar{g} \cdot (\bar{h} \cdot \bar{z}) = \bar{g}\,\alpha(h,z)\overline{hz} = \overline{ghz}\,\alpha(h,z)\,\alpha(g,hz),$$

and since $\alpha(g,h)\,\alpha(gh,z) = \alpha(h,z)\,\alpha(g,hz)$ we have

$$(\bar{g} \cdot \bar{h}) \cdot \bar{z} = \bar{g} \cdot (\bar{h} \cdot \bar{z}).$$

In particular, $F^{\alpha}[G]$ has the unit $\nu\bar{1}$, where $\nu = \alpha(1,1)^{-1}$. That is, for all $g \in G$

$$(\nu\bar{1})\bar{g} = \nu(\overline{1g})\alpha(1,g) = \bar{g}\nu\alpha(1,g) = \bar{g}$$
$$\bar{g}(\nu\bar{1}) = \nu(\overline{g1})\alpha(g,1) = \bar{g}\,\nu\alpha(g,1) = \bar{g}.$$

For each $g \in G$, $\bar{g}$ has the inverse $\overline{g^{-1}}\alpha(g,g^{-1})^{-1}\nu = \overline{g^{-1}}\alpha(g^{-1},g)^{-1}\nu$ in $F^{\alpha}[G]$, because of that

$$\bar{g} \cdot \overline{g^{-1}}\,\alpha(g,g^{-1})^{-1}\nu = \bar{1}\nu.$$

Moreover, the set of A-factor sets forms a group under pointwise multiplication, that is, for any two A-factor sets α and β

$$\alpha \cdot \beta(x,y) = \alpha(x,y)\beta(x,y).$$

Then $\alpha\cdot\beta(xy,z)\alpha\cdot\beta(x,y)=\alpha\cdot\beta(x,yz)\,\alpha\cdot\beta(y,z)$ and

$$\begin{aligned}\alpha^{-1}(x,y)=(\alpha(x,y))^{-1}\Longrightarrow \alpha^{-1}(xy,z)\alpha^{-1}(x,y)&\\ =[\alpha(xy,z)\alpha(x,y)]^{-1}&\\ =[\alpha(x,yz)\alpha(y,z)]^{-1}&\\ =\alpha^{-1}(x,yz)\alpha^{-1}(y,z)&\end{aligned}$$

is the inverse of $\alpha(x,y)$, where $x,y,z\in G$.

If $\mu : G \longrightarrow A$ an arbitrary function, we define $\delta(\mu): G\times G \longrightarrow A$ by

$$\delta(\mu)(x,y)=\mu(x)\mu(y)\mu(xy)^{-1}.$$

Then $\delta(\mu)$ an A-factor set, that is ,

$$\begin{aligned}\delta(\mu)(xy,z)\delta(\mu)(x,y)&=\mu(xy)\mu(z)\mu(xyz)^{-1}\mu(x)\mu(y)\mu(xy)^{-1}\\ &=\mu(x)\mu(y)\mu(z)\mu(xyz)^{-1}\end{aligned}$$

and

$$\begin{aligned}\delta(\mu)(x,yz)\delta(\mu)(y,z)&=\mu(x)\mu(yz)\mu(xyz)^{-1}\mu(y)\mu(z)\mu(yz)^{-1}\\ &=\mu(x)\mu(y)\mu(z)\mu(xyz)^{-1}.\end{aligned}$$

For two A-factor sets α and β, if there exists a function $\mu : G \longrightarrow A$ such that $\beta=\alpha\delta(\mu)$, then we say that α and β are *equivalent* ([1], [2]).

PROPOSITION 2. (1) *If $F^{\times}$-factor sets α and β are equivalent, then $F^{\alpha}[G]\cong F^{\beta}[G]$.*

(2) *For an A-factor set α and $g\in G$,*

$$\alpha(x,y)=\alpha(y,x) \qquad if \quad x,y\in <g>.$$

Proof. (1) Let $\{\overline{g}_{\alpha}\mid g\in G\}$ be a basis of $F^{\alpha}[G]$ and $\{\overline{g}_{\beta}\mid g\in G\}$ a basis of $F^{\beta}[G]$. We define an F-homomorphism

$$f: F^{\alpha}[G]\longrightarrow F^{\beta}[G]$$

by $f(\overline{g}_\alpha) = \overline{g}_\beta \mu(g)^{-1}$, where $\beta = \alpha\delta(\mu)$. Then $f(\overline{g}_\alpha \cdot \overline{h}_\alpha) = f(\overline{g}_\alpha)f(\overline{h}_\alpha)$ because of that $\overline{g}_\alpha \cdot \overline{h}_\alpha = \overline{(gh)}_\alpha \alpha(g,h)$ implies that

$$\begin{aligned} f(\overline{g}_\alpha \cdot \overline{h}_\alpha) &= f(\overline{(gh)}_\alpha \alpha(g,h)) \\ &= \alpha(g,h) f(\overline{(gh)}_\alpha) \\ &= \alpha(g,h)\overline{(gh)}_\beta \ \mu(gh)^{-1} \end{aligned}$$

$$\begin{aligned} f(\overline{g}_\alpha)f(\overline{h}_\alpha) &= \overline{g}_\beta \ \mu(g)^{-1}\overline{h}_\beta \ \mu(h)^{-1} \\ &= \overline{(gh)}_\beta \ \beta(g,h)\mu(g)^{-1}\mu(h)^{-1} \\ &= \overline{(gh)}_\beta \ \mu(g)\mu(h)\mu(gh)^{-1}\mu(g)^{-1}\mu(h)^{-1}\alpha(g,h) \\ &= \alpha(g,h)\overline{(gh)}_\beta \ \mu(gh)^{-1}. \end{aligned}$$

The inverse of f is defined by $f^{-1}(\overline{g}_\beta) = \overline{g}_\alpha \ \mu(g)$,

$$f^{-1} : F^\beta[G] \longrightarrow F^\alpha[G].$$

Thus $F^\alpha[G] \cong F^\beta[G]$.

(2) We assume that $x = g^m$, $y = g^n$, $m \geq n$, $m - n = l$. Since $x = g^l \cdot g^n = g^n \cdot g^l$,

$$\begin{aligned} \alpha(x,y)\alpha(g^n,g^l) &= \alpha(g^n \cdot g^l, g^n)\alpha(g^n,g^l) \\ &= \alpha(g^n, g^l \cdot g^n)\alpha(g^l,g^n) \\ &= \alpha(y,x)\alpha(g^l,g^n). \end{aligned}$$

Now,when $l = 0$ it holds that $\alpha(g^n,1) = \alpha(1,g^n)$ by Lemma 1. Thus we may suppose that

$$\alpha(g^n,g^l) = \alpha(g^l,g^n)$$

Therefore,we have $\alpha(x,y) = \alpha(y,x)$ □

We shall next describe the cohomology theory of G. For an abelian group (with multiplication), a *crossed homomorphism* of G to A is a function $f : G \longrightarrow A$ such that

$$f(xy) = xf(y)f(x), \qquad \text{for all} \quad x,y \in G,$$

where A is a G-module. Then $x = 1 = y$ implies $f(1) = 1$ (identity of A). If A is a trivial G-module (i.e., $xa = a$, for all $x \in G$, $a \in A$), then a crossed homomorphism is just an ordinary homomorphism. The set of all crossed homomorphism of G to A is an abelian group which is denoted by $Z^1_\varphi(G, A)$, where $\varphi : G \longrightarrow A$ is the G-module structure of A ([4]). For each fixed $a \in A$, the function $f_a(x) = xa \cdot a^{-1}$ (for all $x \in G$) is a crossed homomorphism which is called a *principal crossed homomorphism.* In this case, we can prove that $f_a \cdot f_b = f_{ab}$ and $f_{a^{-1}} = (f_a)^{-1}$ ([4]). We denote $B^1_\varphi(G, A)$ the set of all principal crossed homomorphisms which is a subgroup of $Z^1_\varphi(G, A)$. Then the first cohomology group of G with coefficients in A is defined by

$$H^1_\varphi(G, A) = Z^1_\varphi(G, A)/B^1_\varphi(G, A),$$

where $\varphi : G \longrightarrow A$ is the G-module structure of A.

The second cohomology group $H^2(G, A)$ of G with coefficients in A is defined as follows.

A given group G acts to any abelian group A such that $xa = a$ for $x \in G$ and $a \in A$. A group extension E of A by G is a short exact sequence

$$E : 1 \longrightarrow A \xrightarrow{\kappa} B \xrightarrow{\sigma} G \longrightarrow 1.$$

To each $x \in G$, choose a representative $u(x)$ in B, that is, $\sigma(u(x)) = x$. Then for $x, y \in G$, there exists a function $\alpha : G \times G \longrightarrow A$ such that

$$u(x)u(y) = \alpha(x, y)u(xy). \tag{2}$$

Since $u(1)$ is the unit of B, it is clear that

$$\alpha(1, y) = \alpha(1, 1) = \alpha(x, 1)$$

where $\kappa(\alpha(1, y)) = u(1)$ is the unit of B. The function α is called a factor set of the extension E. Since for $x, y, z \in G$,

$$(u(x)u(y))u(z) = (\alpha(x, y)u(xy))u(z) = \alpha(x, y)\alpha(xy, z)u(xyz)$$
$$u(x)(u(y)u(z)) = u(x)(\alpha(y, z)u(yz)) = \alpha(x, yz)\alpha(y, z)u(xyz)$$

we have the following:

$$\alpha(xy,z)\alpha(x,y)=\alpha(x,yz)\alpha(y,z). \tag{3}$$

The factor set α for an extension depends on a choice of representatives: if $u'(x)$ is a second set of representatives with the unit $u'(1)$ of B, then $u(x)$ and $u'(x)$ lie in the same coset. So there is a function $\mu : G \longrightarrow A$ with the unit $\mu(1)$ of B such that

$$u'(x)=\mu(x)u(x).$$

Then

$$\begin{aligned} u'(x)u'(y) &= \mu(x)\mu(y)u(x)u(y) \\ &= \mu(x)\mu(y)\alpha(x,y)u(xy) \\ &= \mu(x)\mu(y)\mu(xy)^{-1}\alpha(x,y)u'(xy) \\ &= (\delta\mu)(x,y)\alpha(x,y)u'(xy) \end{aligned}$$

where $(\delta\mu)(x,y)=\mu(x)\mu(y)\mu(xy)^{-1}$ which is a factor set. Thus $u'(x)$ has a new factor set $(\delta\mu)\alpha$ of E. We put $Z^2(G,A)$ the set of all factor sets satisfying (2) and (3), and $B^2(G,A)$ the set of all factor sets with the form $\delta\mu$. Then $H^2(G,A)$ is defined by

$$H^2(G,A)=Z^2(G,A)/B^2(G,A)$$

(cf. [4]).

Thus an A-factor set α is an element of $Z^2(G,A)=Z(G,A)$ and $\delta\mu$ is in $B^2(G,A)=B(G,A)$. In consequence, the cohomology theory of a group G coefficients in A and the theory of projective representations of G to A are closely related to each other.

We say that two A-factor sets of G are *equivalent* if they are congruent mod $B(G,A)$. Thus, $H^2(G,A)=H(G,A)$ is the set of equivalence class of A-factor sets on G. If $\mathfrak{X}$ is a projective F-representation on G with factor set α, $\mu : G \longrightarrow F^\times$ is any function, and $\mathfrak{Y}=\mathfrak{X}\mu$ is defined by

$$\mathfrak{Y}(g)=\mathfrak{X}(g)\mu(g),$$

then $\mathfrak{Y}$ is a projective representation of G with factor set $\beta = \alpha\delta(\mu)$. Thus $\mathfrak{X}$ and $\mathfrak{Y}$ have equivalent factor sets.

A *central extension* of G is a (possibly infinite) group Γ together with a homomorphism π of Γ onto G such that $\operatorname{Ker}\pi \subseteq Z(\Gamma)$ (the center of Γ). In particular, for an abelian group A and for $\alpha \in Z(G, A)$, there exists a central extension (Γ, π) of G with $\operatorname{Ker}\pi = A$ and such that a set of coset representatives $\{x_g \mid g \in G\}$ exists with $\pi(x_g) = g$ and $x_g x_h = \alpha(g, h)x_{gh}$ (cf. [1]). In this case, Γ is defined such that $\Gamma = G \times A$ with multiplication

$$(g, a)(h, b) = (gh, \alpha(g, h)ab)$$

(cf. [1]). Since $(1, z)(g, a) = (g, \alpha(1, g)za) = (g, a)$, $z = \alpha(1, 1)^{-1}$, and also

$$(g^{-1}, a^{-1}\alpha(g^{-1}, g)^{-1}z)(g, a) = (1, z),$$

we have the identity $1 = (1, z)$ in Γ (cf. [1]).

For $\alpha \in Z(G, A)$ (A is an abelian) $g \in G$ is α-***special*** if $\alpha(g, c) = \alpha(c, g)$ for all $c \in C_G(g) = \{h \in G \mid gh = hg\}$.

PROPOSITION 3. *Let $(\Gamma, \pi) = (G \times A, \pi)$ be the central extension of G such that $\operatorname{Ker}\pi = A$. Then for $x \in \Gamma$, $\pi(x)$ is α-special if and only if $\pi(C_\Gamma(x)) = C_G(\pi(x))$.*

Proof. Assume that $\pi(x)$ is α-special. Then for all $h \in C_G(\pi(x))$ $(x = (g, a))$,

$$\alpha(g, h) = \alpha(h, g),$$

where $\pi(y) = \pi(h, b) = h$. Thus

$$(g, a)(h, b) = (gh, \alpha(g, h)ab)$$
$$(h, b)(g, a) = (hg, \alpha(h, g)ab)$$

imply $y = (h, b) \in C_\Gamma(x)$. Thus $C_G(\pi(x)) \subset \pi(C_\Gamma(x))$. Since $\pi(C_\Gamma(x)) \subset C_G(\pi(x))$, we have

$$C_G(\pi(x)) = \pi(C_\Gamma(x)).$$

Conversely, we assume that $C_G(\pi(x)) = \pi(C_\Gamma(x))$. For each $h \in C_G(\pi(x))$, we have

$$\begin{aligned}(g,a)(h,b) &= (gh, \alpha(g,h)ab) \\ &= (h,b)(g,a) \\ &= (hg, \alpha(h,g)ab) \qquad \text{in} \quad C_\Gamma(x),\end{aligned}$$

and thus $\pi(x)$ is α-special. $\square$

THEOREM 4. *In the central extension* $(\Gamma.\pi) = (G \times A, \pi)$, *for* $\alpha \in Z(G,A)$, *let* $g \in G$ *be* α-*special. Then every conjugate of* g *in* G *is also* α-*special.*

Proof. For all $g^h = hgh^{-1}$ $(g, h \in G)$, we have to prove that

$$C_G(hgh^{-1}) = \pi(C_\Gamma(x)),$$

where $x = (hgh^{-1}, a) = (g^h, a)$ $(a \in A)$. Notice that

$$g^h c = cg^h \implies g^h = g^{ch} \implies h = ch \implies c = 1 \quad (c \in G),$$

$$g^h c^h = c^h g^h \implies (gc)^h = (cg)^h \implies cg = gc,$$

and

$$g^h c^k \neq c^k g^h \qquad \text{if} \quad h \neq k \quad \text{in} \quad G.$$

For $c^h \in C_G(g^h)$, it is suffices to prove that for $y = (c^h, b) \in C_\Gamma(x)$ $(x = (g^h, a))$, $xy = yx$. We note that $c \in C_G(g)$ and $\alpha(g,c) = \alpha(c,g)$ since g is α-special. So,

$$x = (g^h, a) = (h^{-1}gh, a) = (h^{-1}, d)(g, e)(h, f),$$

where d, e, f are determined by

$$(h^{-1}gh, a) = (h^{-1}gh, \alpha(h^{-1}g, h)\alpha(h^{-1}, g)def).$$

Similarly,

$$y = (c^h, b) = (h^{-1}d')(c, e')(h, f').$$

Therefore,

$$xy = (h^{-1}, d)(g, e)(h, f)(h^{-1}, d')(c, e')(h, f').$$

Since,

$$(h, f)(h^{-1}, d') = (1, \alpha(h, h^{-1})fd') \in Z(\Gamma),$$

we have that

$$\begin{aligned} xy &= (h^{-1}, d)(g, e)(c, e')(h, f)(h^{-1}, d')(h, f') \\ &= (h^{-1}, d)(c, e')(g, e)(h, f)(h^{-1}, d')(h, f') \\ &= (h^{-1}, d')(c, e')(h, f')(h^{-1}, d)(g, e)(h, f) \\ &= yx. \end{aligned}$$

Thus

$$c^h = \pi(y) \quad \text{and} \quad y \in C_\Gamma(x).$$

That is,

$$C_G(g^h) = \pi(C_\Gamma(x)).$$

This implies that g^h is α-special by Proposition 3. □

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