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A REMARK ON $G_n(X)$

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1. Introduction

The Gottlieb group $G_n(X)$ of a connected topological space X consists of all $\alpha \in \pi_n(X, x_0)$ such that there is an associated map $A : S^n \times X \to X$ and a homotopy commutative diagram

$$\begin{array}{ccc} S^n \times X & \xrightarrow{A} & X \\ \uparrow & \nearrow \alpha \vee 1_X & \\ S^n \vee X & & \end{array}$$

This group $G_n(X)$ is also characterized by $G_n(X)$ $=w_\sharp(\pi_n(Map(X), 1_X)) \subset \pi_n(X, x_0)$, where $Map(X)$ is the set of all continuous maps from X into itself and $w : Map(X) \to X$ is the evaluation map at $x_0 \in X$, that is, $w(f) = f(x_0)$ with the compact open topology. Thus $G_n(X)$ is also called evaluation subgroup of $\pi_n(X, x_0)$. Gottlieb extensively studied the nature of $G_1(X)$ in [2] and various properties of $G_n(X), n \geq 2$, in [3]. On the other hand, B. J. Jiang is the first person to recognize the importance of $G_1(X) \subset \pi_1(X, x_0)$ in the study of fixed point theorems in [4]. If $G_1(X) = \pi_1(X, x_0)$, then for any continuous map $f : X \to X$ all the Nielsen fixed point classes F of f have the same fixed point index $i(F)$. If we denote this common index by $i(f)$, then we have $L(f) = i(f)N(f)$, where $L(f)$ and $N(f)$ are the Lefschetz

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number and the Nielsen number of f respectively. Thus we were able to compute the seemingly hard to compute Nielsen numbers via Lefschetz numbers and fixed point indexes when X satisfies $G_1(X) = \pi_1(X, x_0)$[1, 4]. It is well known that H-spaces and lens spaces satisfy this condition. Thus it is an interesting problem to find out what kind of spaces satisfy this condition. Recently, Pak has shown that the total spaces of principal bundle over simply connected space satisfy this condition [5]. Then via the Steenrod classification of principal bundles over n-spheres $S^n, n \geq 2$ we can find an infinitely many new examples of spaces which satisfy this condition.

The purpose of this short paper is to generalize the above result. That is, if E is a total space of a principal G-bundle over n-connected space with fiber F then $G_n(E) = \pi_n(E, e) = i_\sharp(\pi_n(G))$. Then our application of this theorem enables us to compute $G_n(X)$ when X is a total space of a bundle over S^n, n-dimensional sphere.

2. Main Theorem

Our theorem is a generalization of a proposition given in [5]. Proof is very similar to that given in [5] but for reader's convenience we prove it here.

THEOREM. *Let $\mathcal{F} = \{E, p, B\}$ be a principal G-bundle over an n-connected space B with $F = p^{-1}(b_0)$ as fiber. Then the total space E satisfies $G_n(E) = \pi_n(E, e) \simeq i_\sharp(\pi_n(G))$.*

Proof. We have the following fiber homotopy exact sequence,

$$\cdots \to \pi_n(F, e) \xrightarrow{i_\sharp} \pi_n(E, e) \xrightarrow{p_\sharp} \pi_n(B, b_0) \to \cdots$$

Since $\pi_n(B, b_0) = 0$, $i_\sharp$ is an onto homomorphism. Since E is a total space of a principal G-bundle, G is a group of homeomorphisms of E. Let $i : G \to E$ be given by the inclusion map $i(g) = g(e)$, where $e \in E$ is the base point. Let $Map(E)$ be the set of all continuous self maps on E into itself and $w_\sharp : Map(E) \to E$ be the evaluation map at $e \in E$. Then i factors through $Map(E)$ and we have the following induced commutative diagram

Since $i_\sharp$ is onto, $w_\sharp$ is onto and $G_n(E) = \pi_n(E, e) = i_\sharp(\pi_n(G, id))$.

REMARK. Note that in our theorem we can replace "principal" with the condition $\pi_n(F) \simeq \pi_n(G)$, where G is a bundle group.

3. Application

We would like to determine $G_3(E_n)$ where E_n is the total space of a S^2 bundle over S^4. According to Steenrod [6], total spaces of S^2 bundles over S^4 are classified by $\pi_3(SO(3)) \simeq Z$. Therefore, for each $n \in Z$ there corresponds a bundle space E_n. Topological classifications are given by $E_n \simeq E_{-n}$. Let $\gamma : SO(3) \to E_n$ be given by $\gamma(e_0)$ at the base point $e_0 \in E_n$ and let $f : \widetilde{E}_n \to E_n$ be the associated principal map. Let $b_0 \in S^4$ be a base point of S^4 such that $p(e_0) = b_0$, and let S^2 and $SO(3)$ be the fibers of E_n and $\widetilde{E}_n$ respectively over $b_0 \in S^4$. Then we have the following commutative diagram of the fiber homotopy exact sequences:

$$\begin{array}{ccccccccc} 0 \to & \pi_4(S^4) & \xrightarrow{\partial} & \pi_3(SO(3)) & \xrightarrow{\tilde{i}_\#} & \pi_3(\tilde{E}_n) & \xrightarrow{\tilde{p}_\#} & \pi_3(S^4) & = 0 \\ & \| & & \gamma_\sharp \downarrow \wr & & f_\sharp \downarrow & & \| & \\ 0 \to & \pi_4(S^4) & \xrightarrow{\partial} & \pi_3(S^2) & \xrightarrow{i_\#} & \pi_3(E_n) & \xrightarrow{p_\#} & \pi_3(S^4) & = 0 \end{array}$$

The top row is the fiber homotopy exact sequence of a principal fiber bundle such that the generator of $\pi_4(S^4)$ goes into n times a generator of $\pi_3(SO(3))$ [6], and $\tilde{i}_\sharp(\pi_3(SO(3))) = Z_n = G_3(\widetilde{E}_n)$ by our theorem. In the bottom row, the bundle group is $SO(3)$ and by the 5 lemma we have $\pi_3(\widetilde{E_n}) = \pi_3(E_n)$ and our theorem again implies

$$G_3(E_n) = i_\sharp\gamma_\sharp(\pi_3(SO(3))) \simeq Z_n = \pi_3(E_n).$$

We also can conclude this from the commutative diagram

$$\begin{array}{ccc} \pi_3(SO(3)) & \xrightarrow{\gamma_\sharp} & \pi_3(E_n) \to 0 \\ \downarrow & \nearrow \tilde{\gamma}_\sharp & \\ \pi_3(Map(E_n, id) & & \end{array}$$

where γ and $\tilde{\gamma}$ are evaluation maps at $e_0 \in E_n$.

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