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ON THE STRONG HOMOLOGY GROUPS

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1. Introduction

Let X be a pointed 0-connected CW-complex given with a filtration

$$X_0 \subset X_1 \subset \cdots \subset X_\alpha \subset \cdots, \ \cup_\alpha X_\alpha = X$$

by subcomplexes.

In 1985, J. T. Lisica and S. Mardešić [6] defined the strong homology groups $H_*^S(C)$ of the inverse system $C = (C_\alpha, f_{\alpha\alpha'}, D)$ of chain complexes $(C_\alpha, \partial)_{\alpha \in D}$. In 1993, L. Mdzinarishivili and E. Spanier [9] have constructed the long exact sequence for the derived functor $\varprojlim_\alpha^{(n)}, n \geq 0$ with respect to the M-S cohomology modules $\hat{H}^*(C; A)$. Algebraic topologists have studied the strong homology groups in the several categories inv-Top, pro-Top and CPH-Top ([5],[7],[8]).

The purpose of this paper is to construct an exact sequence for the negative dimensional (r-stage) strong homology groups $H_*^s(C)$ and the derived functor $\varprojlim_\alpha^{(n)}, n \geq 0$. The definition and some results of the derived functor can be found in ([3],[10]).

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2. (r-stage) strong homology groups of inverse systems

Let $C = (C_\alpha, f_{\alpha\alpha'}, D)$ be an inverse system of chain complexes (C_α, ∂) and chain maps $f_{\alpha\alpha'} : C_{\alpha'} \to C_\alpha, \alpha \leq \alpha'$ in derected set D. Let $D^n, n \geq 0$ denote the set of all non-decreasing n-sequence $a = (\alpha_0, \alpha_1, \cdots, \alpha_n), \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$, in D. The $(n-1)$-sequence $(\alpha_0, \cdots, \hat{\alpha_j}, \cdots, \alpha_n) \in D^{n-1}$ is obtained from $a = (\alpha_0, \cdots, \alpha_n)$, by deleting the j-th factor $\alpha_j, 0 \leq j \leq n$.

A *strong p-chain* of $C, p \in \mathbb{Z}$ is a function $x = (\cdots, x_{(\alpha_0,\cdots,\alpha_n)}, \cdots)$, which assigns to every $a = (\alpha_0, \cdots, \alpha_n) \in D^n, n \geq 0$, a $(p+n)$-chain $x_{(\alpha_0,\cdots,\alpha_n)}$ of $(p+n)$-chain group $(C_{(\alpha_0,\cdots,\alpha_n)})_{p+n}$, where $C_{(\alpha_0,\cdots,\alpha_n)}$
$= C_{\alpha_0}$ and $\mathbb{Z}$ denotes the set of all integers. Putting

$$(x+y)_{(\alpha_0,\cdots,\alpha_n)} = x_{(\alpha_0,\cdots,\alpha_n)} + y_{(\alpha_0,\cdots,\alpha_n)},$$

we obtain the strong p-chain groups $C_p^s, p \in \mathbb{Z}$, of the inverse system $C = (C_\alpha, f_{\alpha\alpha'}, D)$ which are defined by

$$C_p^s = \prod_{n=0}^{\infty} \prod_{(\alpha_0,\cdots,\alpha_n)\in D^n} (C_{(\alpha_0,\cdots,\alpha_n)})_{p+n},$$

where $(C_{(\alpha_0,\cdots,\alpha_n)})_{p+n}$ is the $(p+n)$-chain group of the first chain complex C_{α_0}. A boundary operator $d_p : C_{p+1}^s \to C_p^s$ was defined by

$$\begin{cases} (d_p(x))_{(\alpha_0)} = \partial(x_{(\alpha_0)}) \text{ for } n = 0 \\ (-1)^n (d_p x)_{(\alpha_0,\cdots,\alpha_n)} = \partial(x_{(\alpha_0,\cdots,\alpha_n)}) - f_{\alpha_0\alpha_1}(x_{(\alpha_1,\cdots,\alpha_n)}) \\ \qquad - \sum_{j=1}^{n} (-1)^j x_{(\alpha_0,\alpha_1,\cdots\hat{\alpha}_j,\cdots,\alpha_n)} \text{ for } n \geq 1, \end{cases}$$

where $x \in C_{p+1}^s$. Note that $d_p \circ d_{p+1} = 0$, $p \geq -n$ so that (C^s, d) is a chain complex. The *p-dimensional strong homology group*, denoted by $H_p^s(C)$, of the inverse system $C = (C_\alpha, f_{\alpha\alpha'}, D)$ is defined by the homology group of this chain complex (C^s, d), *i.e.*,

$$H_p^s(C) = \ker(d_{p-1} : C_p^s \to C_{p-1}^s)/\mathrm{im}(d_p : C_{p+1}^s \to C_p^s).$$

For a given above chain complex (C^s, d), we can define a new chain complex $(C^{s(r)}, d^r), r \geq 0$ whose p-dimensional chain group, called *r-stage strong p-chain group* $C_p^{s(r)}$, is defined by

$$C_p^{s(r)} = \prod_{n=0}^{r} \prod_{(\alpha_0,\cdots,\alpha_n)\in D^n} (C_{(\alpha_0,\cdots,\alpha_n)})_{p+n},$$

and the boundary operator $d_p^{(r)} : C_{p+1}^{s(r)} \to C_p^{s(r)}$ is obtained by restricting the boundary operator $d_p : C_{p+1}^s \to C_p^s$ to $C_{p+1}^{s(r)} = C_{p+1}^{s(r)} \times 0 \subseteq C_{p+1}^s$. As the special case of the p-dimensional strong homology group $H_p^s(C)$, we have the p-dimensional homology group, denoted by $\bar{H}_p^s(C^{s(r)})$, of the restricted chain complex $(C^{s(r)}, d^r)$ and we also have the homomorphisms $j_{p*}^{r,r+1} : \bar{H}_p^s(C^{s(r+1)}) \to \bar{H}_p^s(C^{s(r)})$ and $j_{p*}^{s(r)} : H_p^s(C) \to \bar{H}_p^s(C^{s(r)})$ induced by the natural projections $j_{p*}^{r,r+1} : C_p^{s(r+1)} \to C_p^{s(r)}$ and $j_{p*}^{s(r)} : C_p^s \to C_p^{s(r)}$ respectively.

DEFINITION 2.1. The *r-stage strong homology group* $H_p^{s(r)}(C)$ is defined by

$$H_p^{s(r)}(C) = j_{p*}^{r,r+1}(\bar{H}_p^s(C^{s(r+1)}) \subseteq \bar{H}_p^s(C^{s(r)}).$$

3. 4-term exact sequence for the negative dimensional (r-stage) strong homology groups

Let $\Gamma : 0 \to A \xrightarrow{a} I \xrightarrow{f} J \to 0$ be an injective resolution of abelian group A. In the previous paper, E. H. Brown[1] shows that there is a pointed 0-connected CW-complex $\hat{B}(I)$ and natural equivalence

$$\hat{\eta}_I : [-, \hat{B}(I)] \to Hom(H_*(-), I).$$

For example, $\hat{B}(I)$ is the Eilenberg MacLane space $K(\pi, n)$ of type (π, n) or the Moore space $M(\pi, n)$ of type (π, n).

We put $P\hat{B}(J) = \{\omega : [0,1] \to \hat{B}(J) \mid \omega(0) = *\}$. $B(\Gamma) = \{(x,\omega) \in \hat{B}(I) \times P\hat{B}(J) \mid \omega(1) = \hat{f}(x)\}$ is said to be the *mapping kernel* of $\hat{f} : \hat{B}(I) \to \hat{B}(J)$. Note that the mapping kernel $B(\Gamma)$ has the structure of H-group. The map $\hat{f} : \hat{B}(I) \to \hat{B}(J)$ makes a Barratt-Puppe sequence ([4]).

$$\cdots \to \Omega B(\Gamma) \xrightarrow{\Omega j_f} \Omega\hat{B}(I) \xrightarrow{\Omega \hat{f}} \Omega\hat{B}(J) \xrightarrow{\varphi_f} B(\Gamma) \xrightarrow{j_f} \hat{B}(I) \xrightarrow{\hat{f}} \hat{B}(J)$$

where $j_f : B(\Gamma) \to \hat{B}(I)$ is defined by $j_f(x,\omega) = x$ and $\varphi_f : \Omega\hat{B}(J) \to B(\Gamma)$ by $\varphi_f(\omega) = (*,\omega)$.

DEFINITION 3.1. For any injective resolution Γ, we define the contravariant functor $CF(-;\Gamma) = [-, B(\Gamma)]$.

LEMMA 3.2. *Let* $\Gamma : 0 \to A \xrightarrow{a} I \xrightarrow{f} J \to 0$ *be an injective resolution . Then the seqence*

$$0 \to Ext(H_*(X), A) \to CF(X;\Gamma) \to Hom(H_{*+1}(X), A) \to 0$$

is exact.

Proof. From the Barratt-Puppe sequence for fibration, we obtain an exact sequence

$$\cdots \to [X, \Omega\hat{B}(I)] \xrightarrow{\Omega \hat{f}_*} [X, \Omega\hat{B}(J)] \xrightarrow{\varphi_{f*}} [X, B(\Gamma)] \xrightarrow{j_{f*}} [X, \hat{B}(I)]$$
$$\xrightarrow{\hat{f}_*} [X, \hat{B}(J)].$$

By the Brown's representability theorem [1] , there is a commutative diagram;

$$\begin{array}{ccc} [X, \hat{B}(I)] & \xrightarrow[\cong]{\hat{\eta}_I} & Hom(H_*(X), I) \\ \hat{f}_* \downarrow & & \tilde{f}_X \downarrow \\ [X, \hat{B}(J)] & \xrightarrow[\cong]{\hat{\eta}_J} & Hom(H_*(X), J) \end{array}$$

It follows that the following sequence is exact

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{coker}(\Omega\hat{f}_*) & \longrightarrow & CF(X;\Gamma) & \longrightarrow & \ker(\hat{f}_*) & \longrightarrow & 0 \\ & & \cong\big\downarrow & & & & \cong\big\downarrow & & \\ & & \mathrm{coker}(\tilde{f}_{SX}) & & & & \ker(\tilde{f}_X) & & \end{array}$$

The hom-ext exact sequence

$$0 \to Hom(H_*(X), A) \xrightarrow{\tilde{a}} Hom(H_*(X), I) \xrightarrow{\tilde{f}_X} Hom(H_*(X), J)$$

$$\to Ext(H_*(X), A) \to 0$$

shows that there are isomorphisms

$$\begin{cases} \ker(\tilde{f}_X) \cong Hom(H_*(X), A) \\ \mathrm{coker}(\tilde{f}_{SX}) \cong Ext(H_*(SX), A). \end{cases}$$

Therefore we obtain the short exact sequence

$$0 \to Ext(H_*(SX), A) \to CF(X;\Gamma) \to Hom(H_*(X), A) \to 0. \quad \square$$

PROPOSITION 3.3. *Let* $\psi : Ext(H_*(X), A) \to \varprojlim_{\alpha}{}^{(0)} \{Ext(H_*(X_\alpha), A)\}$ *be a homomorphism induced by* $i_\alpha : X_\alpha \hookrightarrow X$. *Then we obtain the following isomorphisms[3];*

(1) *ker* $(\psi) \cong \varprojlim_{\alpha}{}^{(1)} \{Hom(H_*(X_\alpha), A)\}$

(2) *coker*$(\psi) \cong \varprojlim_{\alpha}{}^{(2)} \{Hom(H_*(X_\alpha), A)\}$

(3) $Hom(H_*(X), A) \cong \varprojlim_{\alpha}{}^{(0)} \{Hom(H_*(X_\alpha), A)\}$.

LEMMA 3.4. *Let $\{X_\alpha\}_{\alpha\in D}$ denote any direct system of subcomplexes of X with $\cup_\alpha X_\alpha = X$ and let A be any abelian group. Then we obtain the following exact sequence;*

$$0 \to \varprojlim_\alpha{}^{(1)}\{Hom(H_*(X_\alpha),A)\} \to CF(X;\Gamma) \to \varprojlim_\alpha{}^{(0)}\{CF(X_\alpha;\Gamma)\}$$

$$\to \varprojlim_\alpha{}^{(2)}\{Hom(H_*(X_\alpha),A)\} \to 0$$

Proof. Consider the commutative diagram;

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
Ext(H_*(X),A) & \xrightarrow{\psi} & \varprojlim_\alpha{}^{(0)}\{Ext(H_*(X_\alpha),A)\} \\
\downarrow & & \downarrow \\
CF(X;\Gamma) & \xrightarrow{\varphi} & \varprojlim_\alpha{}^{(0)}\{CF(X_\alpha;\Gamma)\} \\
\downarrow & & \downarrow \\
Hom(H_{*+1}(X),A) & \xrightarrow{\cong} & \varprojlim_\alpha{}^{(0)}\{Hom(H_{*+1}(X_\alpha),A)\} \\
\downarrow & & \downarrow \\
0 & & \varprojlim_\alpha{}^{(1)}\{Ext(H_*(X_\alpha),A)\} \\
 & & \downarrow \\
 & & \vdots
\end{array}$$

The first column is exact by Lemma 3.2 and the second column is also exact by the long exact sequence with respect to the derived functor $\varprojlim_{\alpha}^{(n)}, n \geq 0$ ([2]). Using the serpent lemma and the Proposition 3.3, we obtain the following;

(1) ker $(\varphi) \cong \varprojlim_{\alpha}^{(1)}\{Hom(H_*(X_\alpha), A)\}$

(2) coker$(\varphi) \cong \varprojlim_{\alpha}^{(2)}\{Hom(H_*(X_\alpha), A)\}$.

Therefore, we complete the proof of this lemma. □

Let $(\tilde{C}_\alpha, \tilde{\partial}), \alpha \in D$, be the chain complex whose p-chain group is defined by

$$(\tilde{C}_\alpha)_p = \begin{cases} Hom(H_*(X_\alpha), A) & \text{for } p = 0 \\ 0 & \text{for } p \neq 0. \end{cases}$$

and the boundary operator $\tilde{\partial}$ is zero. For $\alpha \leqq \alpha'$, let $\tilde{f}_{\alpha\alpha'} : \tilde{C}_{\alpha'} \to \tilde{C}_\alpha$ denote the chain map derived from the homomorphism $Hom(H_*(X_{\alpha'}), A) \to Hom(H_*(X_\alpha), A)$. Then we have an inverse system $\tilde{C} = (\tilde{C}_\alpha, \tilde{f}_{\alpha\alpha'}, D)$ of chain complexes $(\tilde{C}_\alpha, \tilde{\partial}), \alpha \in D$.

DEFINITION 3.5. The above chain complex $(\tilde{C}_\alpha, \tilde{\partial}), \alpha \in D$, all of whose chain group vanish, except for the 0-dimensional chain group, is said to be the *single chain complex.*

LEMMA 3.6. *If $\tilde{C} = (\tilde{C}_\alpha, \tilde{f}_{\alpha\alpha'}, D)$ is an inverse system of single chain complexes. then we have*

$$\varprojlim_{\alpha}^{(n)}\{Hom(H_*(X_\alpha), A)\} \cong H^s_{-n}(\tilde{C})$$

for all integer n.

Proof. As a special case of the strong p-chain group of $C = (C_\alpha, f_{\alpha\alpha'}, D)$, we have that the strong $(-n)$-chain group of $\tilde{C} = (\tilde{C}_\alpha, \tilde{f}_{\alpha\alpha'}, D)$ is

$$\tilde{C}^s_{-n} = \begin{cases} \prod_{(\alpha_0,\alpha_1,\cdots,\alpha_n)\in D^n} Hom(H_*(X_{(\alpha_0,\alpha_1,\cdots,\alpha_n)}), A) & \text{for } n \geq 0 \\ 0 & \text{for } n < 0, \end{cases}$$

where $Hom(H_*(X_{(\alpha_0,\alpha_1,\cdots,\alpha_n)}),A) = Hom(H_*(X_{\alpha_0}),A)$. A boundary operator $d_{-(n+1)} : \tilde{C}^s_{-n} \to \tilde{C}^s_{-(n+1)}$ is obtained as following;

$(-1)^n(d_{-(n+1)}x)_{(\alpha_0,\alpha_1,\cdots,\alpha_n)} = \partial(x_{(\alpha_0,\alpha_1,\cdots,\alpha_n)}) - f_{\alpha_0\alpha_1}(x_{(\alpha_1,\cdots,\alpha_n)})$

$-\sum_{j=1}^n(-1)^j x_{(\alpha_0,\alpha_1,\cdots,\hat{\alpha}_j,\cdots,\alpha_n)} = -f_{\alpha_0\alpha_1}(x_{(\alpha_1,\cdots,\alpha_n)}) - \sum_{j=1}^n(-1)^j$ $x_{(\alpha_0,\alpha_1,\cdots,\hat{\alpha}_j,\cdots,\alpha_n)} = (-1)^{n+1}(\delta^n x)_{(\alpha_0,\alpha_1,\cdots,\alpha_n)}$, where $\delta^n, n \geq 0$ is the coboundary operator defined in the paper ([3]). Thus we have the proof of this lemma. □

THEOREM 3.7. *If $\tilde{C} = (\tilde{C}_\alpha, \tilde{f}_{\alpha\alpha'}, D)$ is an inverse system of single chain complexes, then we have that the sequence*

$$0 \to H^s_{-1}(\tilde{C}) \to CF(X;\Gamma) \to \varprojlim_\alpha{}^{(0)}\{CF(X_\alpha;\Gamma)\} \to H^s_{-2}(\tilde{C}) \to 0$$

is exact.

Proof. By Lemma 3.4 and Lemma 3.6, we obtain this theorem. □

In the previous paper[8], the sequence

$$0 \to \varprojlim_r{}^{(1)}\{H^{s(r)}_{p+1}(\tilde{C})\} \to H^s_p(\tilde{C}) \to \varprojlim_r{}^{(0)}\{H^{s(r)}_p(\tilde{C})\} \to 0$$

is exact for every integer p. Using this fact, we easily have the following corollary;

COROLLARY 3.8. *If $\varprojlim_r{}^{(1)}\{H^{s(r)}_{p+1}(\tilde{C})\} = 0$ for all p, then the sequence*

$$0 \to \varprojlim_r{}^{(0)}\{H^{s(r)}_{-1}(\tilde{C})\} \to CF(X;\Gamma) \to \varprojlim_\alpha{}^{(0)}\{CF(X_\alpha;\Gamma)\} \to$$

$$\varprojlim_r{}^{(0)}\{H^{s(r)}_{-2}(\tilde{C})\} \to 0$$

is also exact. □

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