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EXTREMAL STRUCTURE OF COMPLEX TOPOLOGICAL SPACE

SUK YOUNG LEE AND DONG SUN SHIN

Dept. of Mathematics,
Ewha Womans University, Seoul 120 - 750, Korea.

1. Introduction

Let U be the unit disk and $\mathcal{A}$ denote the complex linear topological space of functions holomorphic in U endowed with the topology of local uniform convergence on the compact subsets of U. Suppose $\mathcal{F} \subset \mathcal{A}$. A function f in $\mathcal{F}$ is called a support point of $\mathcal{F}$ if f maximizes ReJ over $\mathcal{F}$ for some continuous linear fuctional J on $\mathcal{A}$ such that ReJ is not constant on $\mathcal{F}$. Let $\bar{co}\mathcal{F}$ denote the closed convex sets containing $\mathcal{F}$.

A function in $\bar{co}\mathcal{F}$ is called an extreme point of $\bar{co}\mathcal{F}$ if $f = tf_1 + (1-t)f_2$ implies $f = f_1 = f_2$ whenever $0 < t < 1$ and $f_1, f_2 \in \bar{co}\mathcal{F}$. We denote by $\mathcal{EF}$ and $supp\mathcal{F}$ the set of extreme points and the set of support points of $\mathcal{F}$, respectively.

The set S of normalized univalent functions is a compact subset of the locally convex space $\mathcal{A}$. In view of the Krein-Milman theorem it is important to identify the extreme points of S, because the solutions to any linear extremal problem over S can be reduced to its solution over the set of extreme points. The solutions to linear extremal problems are the support points of S.

It is not known whether every extreme point is a support point or whether every support point is an extreme point. The problem of describing the set of support points of the class S is open and a full solution seems quite difficult.

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In this paper we observe the extremal and topological structure of two dimensinal complex topological space. We give a geometric characterization of extremal functions in the class S of normalized univalent functions in the unit disk. We investigate the Pearce's result on extreme points of S and support points of S with some remarks.

Finally we study the extremal structure of the class of Spirallike functions $S_p(\alpha)$ and support points of $S_p(\alpha)$

2. Characterization of extremal functions in the class S

A particular subset of complex topological space is the class S. This class consists of all functions f which are univalent in U and normalized so that $f(0) = 0$ and $f'(0) = 1$.

It is well known [17] that all rotations of Koebe function, $k_\theta(z) = \frac{z}{(1+e^{i\theta}z)^2}$ are support points of S as well as extreme points of the closed convex hull of S. These functions map the unit disk onto the complement of a radial slit from $\frac{e^{-i\theta}}{4}$ to infinity. A natural question to ask is which of the geometric properties of the function k_θ are typical of those of arbitrary support points of S. If f is a support point of S, then $\Gamma = \mathbb{C}\backslash f(U)$ is a single analytic arc which tends to infinity with increasing modulus and Γ possesses the $\frac{\pi}{4}$-property: the angle between the radius and tangent vectors never exceeds $\frac{\pi}{4}$ in absolute value.

In order to characterize the geometric property of f which is a support point of S. Let J be a continuous linear functional on $\mathcal{A}$ which is not constant on S and let f maximize ReJ on S. If f does not have the monotonic modulus property, it must have the form

$$f = tf_1 + (1-t)f_2, \quad 0 < t < 1$$

where f_1 and f_2 are functions in S which omit open sets. Since the support points have dense range [8, p 285], neither f_1 nor f_2 is a support point of S. Thus

$$ReJ(f_j) < ReJ(f), \quad j = 1, 2$$

By the linearity of J, this implies that

$$ReJ(f) = tReJ(f_1) + (1-t)ReJ(f_2) < ReJ(f)$$

This contradiction shows that f has monotonic modulus property.

Since a continuous linear functional J is its own Frechet differential, it is known [8, p 306] that $\Gamma = \mathbb{C}\backslash f(U)$ consists of finitely many analytic arcs satisfying the differential equation

$$\frac{1}{w^2} J(\frac{f^2}{f-w}) dw^2 > 0 \tag{2.1}$$

Choose a point $w \in \Gamma$, not an endpoint, and consider the function $g = \frac{wf}{w-f}$. Note that g belongs to S and maps the disk U onto the complement of two disjoint arcs extending to ∞. Thus g is not a support point, and so

$$ReJ(g) < ReJ(f)$$

Since J is linear, this inequality is equivalent to

$$ReJ(\frac{f^2}{f-w}) > 0, \quad w \in \Gamma \tag{2.2}$$

where w is not the end point of Γ. Let J be a continuous complex valued functional, linear or not, on the space $\mathcal{A}$.

Consider the problem of finding the maximum of $ReJ(f)$ for all $f \in S$ where the maximum is attained. The fact that $J(\frac{f^2}{f-w}) \neq 0$ assures that the quadratic differential has no singularities on Γ, except perhaps at the endpoints, so that Γ has no corners. In other words, Γ is a single analytiv arc. The inequality (2.2) may be combined with (2.1) to show that

$$Re(\frac{dw}{w})^2 > 0 \text{ on } \Gamma \tag{2.3}$$

which is equivalent to the $\frac{\pi}{4}$-property. We have shown that the angle $arg\{\frac{dw}{w}\}$ between the tangent line and radial line is less than

$\frac{\pi}{4}$ in magnitude everywhere on Γ except perhaps at the tip. Schiffer [16] gave another proof of the associated quadratic differential has a simple pole at ∞. Using the Schiffer's result that $L(f^2) \neq 0$, we can show that the omitted arc is asymptotic to a line at ∞.

Let J be a continuous linear functional on $\mathcal{A}$ which is not constant on S, and let f maximize ReJ on S. Then the arc Γ omitted by f is asymptotic to the half line

$$w = \frac{1}{3}\frac{J(f^3)}{J(f^2)} - J(f^2)t, \qquad t \geq 0. \tag{2.4}$$

at ∞. Moreover the radial angle of Γ tends to zero at infinity. In order to prove the above assertion, let Γ be parametrized by $w = w(t)$, $0 < t < \infty$, in such a way that $w(t) \to \infty$ as $t \to 0$ and the differential equation (2.1) takes the form

$$\frac{1}{w^2}J(\frac{f^2}{f-w})(\frac{dw}{dt})^2 = 1. \tag{2.5}$$

Because $J(f^2) \neq 0$, the substitution $w = v^{-2}$ tranforms Γ to an analytic curve.

$$v = b_1 t + b_3 t^3 + \dots$$

through the origin which satisfies

$$-4J(\frac{f^2}{1-fv^2})(\frac{dv}{dt})^2 = 1$$

or

$$(c_0 + c_1 b_1^2 t^2 + \dots)(b_1^2 + 6b_1 b_3 t^2 + \dots) = -\frac{1}{4},$$

where $c_n = J(f^{n+2})$, $n = 0, 1, 2\dots$ Equating coefficients, we obtion

$$c_0 b_1^2 = -\frac{1}{4}, \qquad c_1 b_1^4 + 6c_0 b_1 b_3 = 0. \tag{2.6}$$

On the other hand,

$$w = v^{-2} = b_1^{-2}t^{-2} - 2b_1^{-3}b_3 + O(t^2), \qquad t \to 0.$$

Thus Γ is asymptotic to the line

$$w = \alpha + \beta t, \qquad t \to \infty$$

where $\alpha = -2b_1^{-3}b_3$ and $\beta = b_1^{-2}$. But the equation (2.6) give

$$b_1^{-2} = -4c_0, \qquad b_1^{-3} = -\frac{c_1}{6c_0}.$$

This proves that Γ approches the half-line (2.4) near infinity. In particular, $arg\{w\} \to arg\{-L(f^2)\}$ as $w \to \infty$ along Γ. Because

$$\Phi(w) = J(\frac{f^2}{f-w}) = -\frac{J(f^2)}{w} + O(\frac{1}{w^2}),$$

it follows that $arg\{\Phi(w)\} \to 0$. Thus the differential equation $\Phi(w)(\frac{dw}{w})^2 > 0$ shows that the radial angle $arg\{\frac{dw}{w}\} \to 0$ as $w \to \infty$ along Γ. Thus comletes the proof.

3. Some remarks on Pearce's result

It has shown [11] that if C denote the class of close-to-convex functions in U, $\bar{co}C$ consists of the functions represented by

$$f(z) = \int_X \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2} d\mu(x,y)$$

where μ varies over the probability measures on $X = \partial U \times \partial U$ and

$$\mathcal{E}\bar{co}C = suppC = \{f(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}; \quad |x| = |y| = 1, \quad x \neq y\}$$

It is well known that $C \subset S$ but we can show that $S \not\subset \bar{co}C$. Since S and $\bar{co}S$ are compact set, a lemma of Dunford and Schwartz [7] shows that in certain cases we can identify support point of S as extreme point of $\bar{co}S$.

LEMMA [7]. *Let J be a continuous linear functional on $\mathcal{A}$ such that ReJ is non-constant on S. If there exist at most two support points of S which ReJ maximize over S, then each support point of S is an extreme point of S.*

It is well known [3] that the Koebe functions

$$k_x(z) = \frac{z}{(1-xz)^2}, \quad |x| = 1$$

uniquely maximize ReJ_x over S, where $J_x g = \bar{x} g''(0)$, $|x| = 1$. Thus, the Koebe functions k_x, $|x| = 1$ are both support points of S and extreme points of $\bar{co}S$.

K.Pearce [14] proved that the functions

$$f(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}, \quad |x| = |y| = 1, \quad x \neq y$$

are support points of S and extreme points of $\bar{co}S$, whenever $0 < |arg(-\frac{x}{y})| \leq \frac{\pi}{4}$. He used Goluzin's result that if $f \in S$, then

$$|arg f'(z_0)| \leq arcsin|z_0|, \quad |z_0| \leq \frac{1}{\sqrt{2}}$$

Pearce used a continuous linear functional $J_{x,y}$ defined by

$$J_{x,y} g = -e^{i(\frac{\pi}{2} - 4arcsin|z_0|)} g'(z_0)$$

when $0 < |arg(-\frac{x}{y})| \leq \frac{\pi}{4}$ and $|z_0| \leq \frac{1}{\sqrt{2}}$. He showed thar $ReJ_{x,y}$ is uniquely maximized over S by $f_{x,y}$ if $0 < |arg(-\frac{x}{y})| \leq \frac{\pi}{4}$
In fact, the function

$$f(z) = \frac{z - \frac{1}{2}2(x+y)z^2}{(1-yz)^2}, \quad |x| = |y| = 1, x \neq y$$

maps the unit disk U one-to-one onto the complement of a ray. To see this, we may assume that $y = 1$, $x \neq 1$. If $a = \frac{1}{2}(1+x)$ then $|a - \frac{1}{2}| = \frac{1}{2}$ and $a \neq 1$. Thus

$$f(z) = \frac{1}{4(1-a)}[\{(1-a)\frac{1+z}{1-z} + a\}^2 - 1].$$

Since $w = \frac{1+z}{1-z}$ maps U one-to-one onto $\{w : Rew > 0\}$ it follows that f maps u onto the complement of a closed ray. The tip of the ray is attained when

$$(1-a)\frac{1+z}{1-z} + a = 0 \text{ i.e. when } z = \frac{1}{2a-1}.$$

We note that

$$f(\frac{1}{2a-1}) = -\frac{1}{4(1-a)}$$

and that since $f(\pm i) = -\frac{1}{2} \pm \frac{1}{2}(ia)$ are on tha ray so is the point $-\frac{1}{2}$. If we let $x = e^{i\theta}$, $0 < \theta < 2\pi$, then

$$-\frac{1}{4(1-a)} = -\frac{1}{4} - \frac{i}{4}cot\frac{\theta}{2}$$

Thus, with varying θ, the rays obtained consists of all rays through $w = -\frac{1}{2}$ and having the tip on the line $Rew = -\frac{1}{4}$. It is evident that if $|cot\frac{\theta}{2}| > 1$, then the ray will not have a strictly increasing modulus. Therefore, if $|\theta| < \frac{\pi}{2}$,then $\Gamma = \mathbb{C}\backslash f(U)$ does not satisfy the fact [1,p59] that if $f \in \mathcal{E}S$, then $\Gamma = \mathbb{C}\backslash f(U)$ is an unbonuded continuous curve having a strictly increasing modulus. So we have shown that if

$$f(z) = \frac{z - \frac{1}{2}(x+1)z^2}{(1-z)^2}, \quad x = e^{i\theta}, \quad (0 < \theta < 2\pi)$$

and if $|\theta| < \frac{\pi}{2}$, then f does not belong to $\mathcal{E}S$. Using this result, we can shown that $S \not\subset \bar{co}C$.

Suppose $S \subset \bar{co}C$. Then $\bar{co}S \subset \bar{co}C$ and this implies that $\mathcal{E}\bar{co}S \subset \mathcal{E}\bar{co}C$. This is impossible because

$$f(z) = \frac{z - \frac{1}{2}(x+1)z^2}{(1-z)^2}, \quad x = e^{i\theta}, \quad (0 < \theta < 2\pi)$$

belongs to $\mathcal{E}\bar{co}C$.

4. Extremal structure of the class of Spirallike functions

An α-spiral is a curve in the complex plane of the form

$$w = w_0 exp(-e^{-i\alpha}t), \ -\infty < t < \infty, \ w_0 \neq 0, \ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}.$$

A domain D containing the origin is said to be α-spirallike if for each point $w_0 \neq 0$ in D the arc of the α-spiral from w_0 to the origin lies entirely in D. If $f(z)$ is analytic and univalent in U, with $f(0) = 0$, it is said to be α-spirallike if its range is α-spirallike. Let $S_p(\alpha)$ denote the class of α-spirallike functions f with parameter α, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$. This class $S_p(\alpha)$ is characterized by the conditions

$$(4.1) \quad f(0) = 0, \ f'(0) = 1 \ \text{ and } \ Re[e^{i\alpha} z \frac{f'(z)}{f(z)}] > 0 \text{ for } |z| < 1$$

We note that $S_p(\alpha) \subset S^*$, the class of starlike functions with respect for the orign and $S_p(\alpha) \subset S$
The determination of $\mathcal{E}\bar{co}S_p(\alpha)$ and $suppS_p(\alpha)$ remains an open problem. In this section we observe the extremal structure of the class $S_p(\alpha)$ to provide some useful information in this direction. It was shown in [18] that the class $S_p(\alpha)$ is a compact subset of locally convex topological space $\mathcal{A}$. Let $\mathcal{P}$ denote the class of functions p with $Rep(z) > 0$ and $p(0) = 1$ in U. It is well known [1,p30] that $p \in \mathcal{P}$ if and only if there is a probability measure μ on ∂U such that

$$(4.2) \qquad p(z) = \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x).$$

THEOREM 4.1. *$f \in S_p(\alpha)$ if and only if there is a probability measure μ on ∂U such that*

$$(4.3) \quad f(z) = zexp[\int_{|x|=1} -2\tau log(1 - xz) d\mu(x)], \ \tau = cos\alpha e^{-i\alpha}$$

The correspondence between the set of probability measures on ∂U and $S_p(\alpha)$ given by (4.3) is one-to-one.

Proof. [18. p 605]

Let Λ denote the set of probability measures on ∂U. The $\mathcal{E}\Lambda$ consists of the point masses [11]. It is known [11, p 31] that $p \in \mathcal{P}$ if and only if there exists a sequence of functions $\{p_n\}$ so that each has the form

$$q(z) = \sum_{k=1}^{m} t_k \frac{1 + x_k z}{1 - x_k z} \tag{4.4}$$

where $|x_k| = 1, t_k \geq 0, \sum_{k=1}^{m} t_k = 1$, and $p_n \longrightarrow p$ uniformly on compact subsets of U.
Using the above informations we can prove that every function $f \in S_p(\alpha)$ can be approximated uniformly on $|z| \leq r$ by functions of the form of finite product;

THEOREM 4.2. *$f \in S_p(\alpha)$ if and only if there is a sequence of functions $\{f_n\}$ having the form*

$$q(z) = \frac{z}{\Pi_{k=1}^{m}(1 - x_k z)^{2\tau t_k}} \tag{4.5}$$

where $|x_k| = 1, t_k \geq 0, \sum_{k=1}^{m} 2\tau t_k = 2\tau, \tau = cos\alpha e^{-i\alpha}, |\alpha| < \frac{\pi}{2}$, and $f_n \longrightarrow f$ uniformly on compact subsets of U.

Proof. According to Theorem 4.1, $f \in S_p(\alpha)$ if and only if

$$\begin{aligned} f(z) &= zexp[\tau \int_0^z \frac{p(\zeta) - 1}{\zeta} d\zeta], \ \tau = cos\alpha e^{-i\alpha} \\ &= zexp[\int_{|x|=1} -2\tau log(1 - xz) d\mu(x)] \end{aligned}$$

Since μ is a probability measure on ∂U, there is a sequence of probability measure ν_n which are convex combinations of point masses so that

$$\int_{|x|=1} g(x) d\nu_n(x) \longrightarrow \int_{|x|=1} g(x) d\mu(x)$$

for every continuous function g on ∂U. We may write

$$\nu_n = \sum_{k=1}^{m} t_k \delta_{x_k}$$

where $t_k \geq 0, \sum_{k=1}^{m} t_k = 1$ and δ_{x_k} denotes point mass at x_k (hence $|x_k| = 1$ for $k = 1, 2, \ldots, m$). If f_n denotes the functions in $S_p(\alpha)$ corresponding to the measures ν_n it follows that $f_n \longrightarrow f$ uniformly on compact subsets of U. The uniform convergence follows from Montel's theorem and the fact that

$$|f_n(z)| \leq max_{|x|=1} \frac{|z|}{|1 - xz|^2\tau}, \quad \tau = cos\alpha e^{-1\alpha}$$

Also, if $f_n \in S_p(\alpha)$, we may write

$$f_n(z) = zexp[\tau \int_1^z \frac{p_n(\zeta) - 1}{\zeta} d\zeta], \ p_n(\zeta) \in \mathcal{P}, \ \tau = cos\alpha e^{-i\alpha}$$

By applying (4.4) for $p_n(\zeta) \in \mathcal{P}$, we have

$$\begin{aligned} f_n(z) &= zexp[\tau \int_0^z \frac{\sum_{k=1}^{m} t_k (\frac{1+x_k\zeta}{1-x_k\zeta} - 1)}{\zeta} d\zeta] \\ &= zexp[2\tau \int_0^z \sum_{k=1}^{m} \frac{t_k x_k}{1 - x_k\zeta} d\zeta] \\ &= zexp[-2\tau \sum_{k=1}^{m} log(1 - x_k z)] \\ &= zexp[log\Pi_{k=1}^{m}(1 - x_k z)^{-2\tau t_k}] \\ &= \frac{z}{\Pi_{k=1}^{m}(1 - x_k z)^{2\tau t_k}} \end{aligned}$$

where $|x_k| = 1, \ t_k \geq 0, \ \sum_{k=1}^{m} 2\tau t_k = 2\tau, \ \tau = cos\alpha e^{-i\alpha}, \ |\alpha| < \frac{\pi}{2}$. The converse follows from the fact that each g given by (4.5) is in $S_p(\alpha)$.

THEOREM 4.3. *If* $f \in S_p(\alpha)$, *then* $\frac{f(z)}{z}$ *is subordinate to*

$$F(z) = \frac{1}{(1-z)^{2\tau}}, \text{ where } \tau = cos\alpha e^{-i\alpha}$$

Proof. If $f \in S_p(\alpha)$, we may write $e^{i\alpha}\frac{zf'(z)}{f(z)} = cos\alpha p(z) + isin\alpha$, where $p(z) \in \mathcal{P}$. Let

$$q(z) = \frac{zf'(z)}{f(z)} = e^{-i\alpha}[cos\alpha p(z) + isin\alpha].$$

Since $f(z) \neq 0$ when $z \neq 0$, we have

$$\frac{d}{dz}[log\frac{f(z)}{z}] = \frac{q(z)-1}{z}$$

and

$$log[\frac{f(z)}{z}] = \int_0^z \frac{q(\zeta)-1}{\zeta}d\zeta \tag{4.6}$$

where $q(\zeta) = e^{-i\alpha}[cos\alpha p(\zeta) + isin\alpha]$.
The function $\{q\}$ consists of those functions subordinate to

$$q_0(\zeta) = e^{-i\alpha}[cos\alpha\frac{1+\zeta}{1-\zeta} + isin\alpha]$$

which is the function mapping U onto a half plane. Also,

$$\begin{aligned}\int_0^z \frac{q_0(\zeta)-1}{\zeta}d\zeta &= \int_0^z \frac{1}{\zeta}[e^{-i\alpha}(cos\alpha\frac{1+\zeta}{1-\zeta} + isin\alpha) - 1]d\zeta \\ &= \int_0^z \frac{2cos\alpha e^{-i\alpha}}{1-\zeta}d\zeta \\ &= -2\tau log(1-z), \text{ where } \tau = cos\alpha e^{-i\alpha}\end{aligned}$$

The integral in (4.6) defines a continuous linear operator of order zero on the family $\{q\}$ and $-2\tau log(1-z)$ is univalent and convex in U. Therefore $\frac{f(z)}{z}$ is subordinate to $F(z) = \frac{1}{(1-z)^{2\tau}}, \tau = cos\alpha e^{-i\alpha}$.

REMARK. *Let $F \in \mathcal{A}$ and $s(F) = \{g \in \mathcal{A} : g \prec F\}$. it was shown in [11] that $\mathcal{E}\bar{co}s(F)$ contains all functions of the form $F(xz)$ with $|x| = 1$. By Theorem 4.3, it is easy to show that $S_p(\alpha) \in zs(F)$, where $F(z) = (1-z)^{-2\tau}, \tau = cos\alpha e^{-i\alpha}$. Hence it is suggested the possibility that*

$$\mathcal{E}\bar{co}s_p(\alpha) = \{\frac{z}{(1-xz)^{2\tau}} : |x| = 1, \tau = cos\alpha e^{-i\alpha}\}$$

However, K.Pearce has shown in [20] that this is not the case because the product theorem does not hold for complex number τ.
If we define a continuous linear functional J_x by

$$J_x g = 2\bar{\tau}\bar{x}g''(0), \quad |x| = 1, \quad \tau = cos\alpha e^{-i\alpha}$$

We can verify that each function

$$f(z) = \frac{z}{(1-xz)^{2\tau}}, \quad |x| = 1, \quad \tau = cos\alpha e^{-i\alpha} \tag{4.7}$$

uniquely maximizes ReJ_x over $S_p(\alpha)$. Hence each function(4.7) is necessarily an extreme point of $\bar{co}S_p(\alpha)$.

LEMMA [6]. *Suppose that $f \in S_p(\alpha)$ and f does not have the form given by*

$$f(z) = \frac{z}{(1-xz)^{2\tau}}, \quad |x| = 1, \quad \tau = cos\alpha e^{-i\alpha}$$

then there are positive number M and ϵ so that

$$|f(z)| \leq \frac{M}{(1-|z|)^{2\tau-\epsilon}}, \quad (|z| < 1) \tag{4.8}$$

THEOREM 4.4. *Let J be a continuous linear functional on $\mathcal{A}$ not of the $J(f) = af(0) + bf'(0)$. The only functions f_0 in $S_p(\alpha)$ that satisfy*

(4.9) $$max\{ReJ(f) : f \in S_p(\alpha)\}$$

are of the form

(4.10) $$f_0 = \sum_{k=1}^{m} t_k f_k$$

where
(4.11)
$$f_k(z) = \frac{z}{(1 - x_k z)^{2\tau}},\ \tau = cos\alpha e^{-i\alpha},\ |x_k| = 1,\ 0 \le t_k \le 1,\ \sum_{k=1}^{m} t_k = 1.$$

Proof. If J is a contiuous linear functional on $\mathcal{A}$, then there is a sequence $\{b_n\}$ of complex numbers satisfying

(4.12) $$\lim_{n\to\infty} sup|b_n|^{\frac{1}{n}} < 1$$

and such that

(4.13) $$J(f) = \sum_{n=0}^{\infty} b_n a_n$$

where$f \in \mathcal{A}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n, (|z| < 1)$.
We have seen that each function

$$f(z) = \frac{z}{(1 - xz)^{2\tau}},\quad |x| = 1,\quad \tau = cos\alpha e^{-i\alpha}$$

is necessarily an extreme point of $\bar{co}S_p(\alpha)$.
since

(4.14) $$f(z) = \frac{z}{(1 - xz)^{2\tau}} = z + \sum_{n=2}^{\infty} \prod_{k=0}^{n-2} \frac{2\tau + k}{k + 1} x^{n-1} z^n$$

the n^{th} coefficient of the power series (4.14) for f(z) is given by $a_n = c_n x^{n-1}$, where

$$c_n = \frac{2\tau(2\tau+1)\dots(2\tau+(n-2))}{(n-1)!}, n = 2, 3\dots$$

Therefore, equation (4.13) implies that

$$J(f) = b_1 + \sum_{n=2}^{\infty} b_n c_n x^{n-1} \tag{4.15}$$

Since $c_n^{\frac{1}{n}} \to 1$ as $n \to \infty$, (4.12) imples that

$$\lim_{n\to\infty} sup|b_n c_n|^{\frac{1}{n}} = \lim_{n\to\infty} sup|b_n|^{\frac{1}{n}} < 1$$

This defines an analytic function $F(x)$ for $|x| \leq 1$, and F is not constant if J does not have the form $J(f) = af(0) + bf'(0)$. The argument given in [3,p100] shows that if F is the set of all functions f_0 in $\bar{co}S_p(\alpha)$ such that

$$ReJ(f_0) = max\{ReJ(F) : f \in \bar{co}S_p(\alpha)\}$$

then there is a positive integer m so that F is the same as the set of functions f_0 given by

$$f_0 = \sum_{k=1}^{m} t_k f_k$$

where

$$f_k(z) = \frac{z}{(1-xz)^{2\tau}},\ \tau = cos\alpha e^{-i\alpha},\ |x_k| = 1,\ 0 \leq t_k \leq 1,\ \sum_{k=1}^{m} t_k = 1.$$

In particular, this shows that the only functions f_0 in $S_p(\alpha)$ that maximize $ReJ(f)$ over $S_p(\alpha)$ have the form prescribed by equations (4.10) and (4.11). In fact, if $t_k \neq 0$ for some k, then by letting z tend to $\bar{x}_k$ radially we see that the function of equation (4.10) does not satisfy the inequality (4.8). Therefore, the only functions given by equations (4.10) and (4.11) in $S_p(\alpha)$ satisfy (4.9). This comletes the proof.

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