Honam Mathematical J. 18(1996), No. 1,pp. 17–23

ON THE GROUP RING $K\mathbb{Z}_2$

WON-SUN PARK

Dept. of Mathematics, Chonnam University, Kwangju 500-757, Korea.

Let K be a field of characteristic 0 and G be a Klein's four group with the fixed order $g_0 = 1, g_1, g_2, g_3$ of elements.

The basic group table matrix of the Klein's four group G with the fixed order $g_0 = 1, g_1, g_2, g_3$ of elements is the symmetric matrix

$$\begin{pmatrix} 1 & g_1 & g_2 & g_3 \\ g_1 & 1 & g_3 & g_2 \\ g_2 & g_3 & 1 & g_1 \\ g_3 & g_2 & g_1 & 1 \end{pmatrix}$$

which is identical with the group table matrix of G.

From the element $\alpha = \sum_{i=0}^{3} r_i g_i$ of the group ring KG, we obtain a following matrix M_α by putting r_i in the place of g_i in the above basic group table matrix of G.

$$M_\alpha = \begin{pmatrix} r_0 & r_1 & r_2 & r_3 \\ r_1 & r_0 & r_3 & r_2 \\ r_2 & r_3 & r_0 & r_1 \\ r_3 & r_2 & r_1 & r_0 \end{pmatrix}$$

Since every nontrivial proper subgroup H of G is a cyclic group of order 2, we have $H \cong \mathbb{Z}_2$ and thus $K\mathbb{Z}_2$ is a subring of KG. For $\beta = r_0 + r_1 g_1 \in K\mathbb{Z}_2$, we have

$$M_\beta = \begin{pmatrix} r_0 & r_1 & 0 & 0 \\ r_1 & r_0 & 0 & 0 \\ 0 & 0 & r_0 & r_1 \\ 0 & 0 & r_1 & r_0 \end{pmatrix}$$

Received May 13, 1996.

In this paper, we shall find the idempotent and nilpotent elements of $K\mathbb{Z}_2$ from KG.

Let

$$P = \begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Then

$$P^{-1}M_\alpha P$$
$$= diag(r_0 + r_1 - r_2 - r_3, r_0 - r_1 + r_2 - r_3, r_0 - r_1 - r_2 + r_3, r_0 + r_1 + r_2 + r_3).$$

Therefore

$$M_\alpha^n$$
$$= \frac{1}{4}\begin{pmatrix} a+b+c+d & a-b-c+d & -a+b-c+d & -a-b+c+d \\ a-b-c+d & a+b+c+d & -a-b+c+d & -a+b-c+d \\ -a+b-c+d & -a-b+c+d & a+b+c+d & a-b-c+d \\ -a-b+c+d & -a+b-c+d & a-b-c+d & a+b+c+d \end{pmatrix}$$

where

$$a = (r_0 + r_1 - r_2 - r_3)^n$$
$$b = (r_0 - r_1 + r_2 - r_3)^n$$
$$c = (r_0 - r_1 - r_2 + r_3)^n$$
$$d = (r_0 + r_1 + r_2 + r_3)^n.$$

LEMMA 1([3] THEOREM 1). *Let K be a field of characteristic 0 and G be a Klein's four group with the fixed order $g_0 = 1, g_1, g_2, g_3$ of elements, then KG has 16 idempotent elements and the values of r_0, r_1, r_2 and r_3 for idempotent $\alpha = \sum_{i=0}^{3} r_i g_i$ of KG are fol-*

lows:

r_0	r_1	r_2	r_3
0	0	0	0
1	0	0	0
$\frac{1}{2}$	0	0	$\pm\frac{1}{2}$
$\frac{1}{2}$	0	$\pm\frac{1}{2}$	0
$\frac{1}{2}$	$\pm\frac{1}{2}$	0	0
$\frac{1}{4}$	$\frac{1}{4}$	$\pm\frac{1}{4}$	$\pm\frac{1}{4}$
$\frac{1}{4}$	$-\frac{1}{4}$	$\pm\frac{1}{4}$	$\mp\frac{1}{4}$
$\frac{3}{4}$	$\frac{1}{4}$	$\pm\frac{1}{4}$	$\mp\frac{1}{4}$
$\frac{3}{4}$	$-\frac{1}{4}$	$\pm\frac{1}{4}$	$\pm\frac{1}{4}$.

From Lemma 1,

$$\begin{aligned}
e_1 &= \frac{1}{4} + \frac{1}{4}g_1 + \frac{1}{4}g_2 + \frac{1}{4}g_3 \\
e_2 &= \frac{1}{4} + \frac{1}{4}g_1 - \frac{1}{4}g_2 - \frac{1}{4}g_3 \\
e_3 &= \frac{1}{4} - \frac{1}{4}g_1 + \frac{1}{4}g_2 - \frac{1}{4}g_3 \\
e_4 &= \frac{1}{4} - \frac{1}{4}g_1 - \frac{1}{4}g_2 + \frac{1}{4}g_3
\end{aligned}$$

are primitive and orthogonal idempotent elements such that $\sum_{i=1}^{4} e_i = 1$.

Therefore we have the following Theorem 1 (1).

THEOREM 1. *Let K be a field of characteristic 0 and G a Klein's four group with the fixed order $g_0 = 1, g_1, g_2, g_3$ of elements. Then*

(1)

$$KG \cong KGe_1 \oplus KGe_2 \oplus KGe_3 \oplus KGe_4$$

where

$$\begin{aligned} e_1 &= \frac{1}{4} + \frac{1}{4}g_1 + \frac{1}{4}g_2 + \frac{1}{4}g_3 \\ e_2 &= \frac{1}{4} + \frac{1}{4}g_1 - \frac{1}{4}g_2 - \frac{1}{4}g_3 \\ e_3 &= \frac{1}{4} - \frac{1}{4}g_1 + \frac{1}{4}g_2 - \frac{1}{4}g_3 \\ e_4 &= \frac{1}{4} - \frac{1}{4}g_1 - \frac{1}{4}g_2 + \frac{1}{4}g_3 \end{aligned}$$

(2) *For the cyclic group $\mathbb{Z}_2$ of order 2, $K\mathbb{Z}_2$ has 4 idempotent elements and $K\mathbb{Z}_2$ is indecomposable.*

Proof. (2) This is the case that $r_2 = r_3 = 0$ in Lemma 1. Hence the values of r_0 and r_1 for the idempotent element $\beta = r_0 + r_1 g_1 \in K\mathbb{Z}_2$ are follows;

r_0	r_1
0	0
1	0
$\frac{1}{2}$	$\pm\frac{1}{2}$.

Since $K\mathbb{Z}_2$ has no primitive and orthogonal idempotent elements, $K\mathbb{Z}_2$ is indecomposable.

LEMMA 2([3] THEOREM 2 AND 3). *Let K be a field of characteristic 0 and G a Klein's four group with the fixed order $g_0 = 1, g_1, g_2, g_3$ of elements. Then*

(1) *If $\alpha = \sum_{i=0}^{3} r_i g_i \in KG$ has a unit such that $\alpha^n = 1$, then α satisfy*

$$\begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \frac{1}{4}\{\rho_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} + \rho_2 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + \rho_3 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + \rho_4 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\}$$

where ρ_1, ρ_2, ρ_3, and ρ_4 are n-th roots of unity in K.

(2) *The number of units* $\alpha = \sum_{i=0}^{3} r_i g_i$ *of* KG *such that* $\alpha^2 = 1$ *is 16 and the values of* r_0, r_1, r_2 *and* r_3 *are following:*

r_0	r_1	r_2	r_3
± 1	0	0	0
0	± 1	0	0
0	0	± 1	0
0	0	0	± 1
$\frac{1}{2}$	$\frac{1}{2}$	$\pm\frac{1}{2}$	$\mp\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{2}$	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$
$-\frac{1}{2}$	$\frac{1}{2}$	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$\pm\frac{1}{2}$	$\mp\frac{1}{2}$.

(3) KG *has no nonzero nilpotent elements.*

(4) $\alpha = \sum_{i=0}^{3} r_i g_i \in KG$ *is a unit if and only if* $r_0 + r_1 \neq \pm(r_2 + r_3)$ *and* $r_0 - r_1 \neq \pm(r_2 - r_3)$.

(5) $\alpha = \sum_{i=0}^{3} r_i g_i \in KG$ *is a 1-unit if and only if* $r_0 + r_1 + r_2 + r_3 = 1$ *and* $r_i + r_j \neq \frac{1}{2}$ $(i \neq j)$.

THEOREM 2. *Let* K *be a field of characteristic 0 and* $\mathbb{Z}_2$ *a cyclic group of order 2. Then*

(1) *If* $\beta = r_0 + r_1 g_1 \in K\mathbb{Z}_2$ *is a unit such that* $\beta^n = 1$. *Then* β *satisfy*

$$\begin{pmatrix} r_0 \\ r_1 \end{pmatrix} = \frac{1}{2}\{\xi_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \xi_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$$

where ξ_1 *and* ξ_2 *are* n*-th roots of unity in* K.

(2) *There are 4 units* β *in* $K\mathbb{Z}_2$ *such that* $\beta^2 = 1$.

(3) $K\mathbb{Z}_2$ *has no nonzero nilpotent elements.*

(4) $\beta = r_0 + r_1 g_1 \in K\mathbb{Z}_2$ *is a unit if and only if* $r_0 \neq \pm r_1$.

(5) $\beta = r_0 + r_1 g_1 \in K\mathbb{Z}_2$ *is a 1-unit if and only if* $r_0 + r_1 = 1$ *and* $r_0 = r_1 \neq \frac{1}{2}$.

Proof. See the case that $r_2 = r_3$ in Lemma 2. That is,

(1)

$$\begin{pmatrix} r_0 \\ r_1 \end{pmatrix} = \frac{1}{4}\{\rho_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \rho_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \rho_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \rho_4 \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$$
$$= \frac{1}{2}\{\xi_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \xi_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$$

where ξ_1 and ξ_2 are n-th roots of unity in K.

(2) The values of r_0 and r_1 of the unit $\beta = r_0 + r_1 \in K\mathbb{Z}_2$ such that $\beta^2 = 1$ are following:

r_0	r_1
± 1	0
0	± 1 .

Let G be a Klein's four group. By considering Lemma 2, we can obtain the units of the group ring $\mathbb{Z}_2 G$. That is, if $\alpha = \sum_{i=0}^{3} r_i g_i \in \mathbb{Z}_2 G$ is a unit, then the values of r_0, r_1, r_2 and r_3 are followings:

r_0	r_1	r_2	r_3
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0 .

Therefore we have the following.

THEOREM 3. *Let G be a Klein's four group. Then the unit group $U\mathbb{Z}_2G$ of $\mathbb{Z}_2G$ is $U\mathbb{Z}_2G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

References

1. V. Bist, *Group of units of group algebra*, Comm. in Algebra **20** (1992), 1747–1761.
2. E. Jespers and G. Leal, *Describing units of intergal group ring of some 2-groups*, Comm. in Algebra **19** (1991), 1809–1827.
3. Won-Sun Park, *On the group ring of the klein's four group*, Comm. Korean Math. Soc. **11** (1996), 63–70.
4. D.S. Passman, *The Algebraic Structure of Group Rings*, J. Weley, New York, 1977.
5. S.K. Sehgal, *Topics in Group Rings*, M. Dekker, New York, 1978.
6. A. Weiss, *Idempotents in group rings*, J. Pure. Appl. Algebra **16** (1972), 207–213.