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PRIME MODULES

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Throughout this paper, unless otherwise stated, R will denote a commutative ring with identity. Let P be a proper ideal of R. Then P is prime if, for any elements a, b in R such that $ab \in P$, either $a \in P$ or $b \in P$. Let Q be a proper ideal of R. Then Q is primary if, for any elements x, y in R such that $xy \in Q$, either $x \in Q$ or there exists $n \in \mathbb{N}$ such that $y^n \in Q$.

It is clear that every prime ideal is primary.

Several authors have extended the notions of prime and primary ideals to modules (see [1]–[8]).

In this paper, we continue these investigations. In [12] Sharp, Tıraş and Yassi introduced concepts of reduction and integral closure of an ideal I of a commutative ring R relative to a Noetherian R–module M, and they showed that these concepts have properties which reflect some of those of the classical concepts of reduction and integral closure introduced by Northcott and Rees in [10]. It is appropriate for us to provide a brief review.

They say that I is a reduction of the ideal J of R relative to a Noetherian R-module M if $I \subseteq J$ and there exists $s \in \mathbb{N}$ such that

$$IJ^sM = J^{s+1}M$$

An element x of R is said to be integrally dependent on I if there exists $n \in \mathbb{N}$ such that $x^nM \subseteq (\sum_{i=1}^{n} x^{n-1}I^i)M$. In fact this is the case if and only if I is a reduction of $I + Rx$ relative to M [12,(1.5)]. Moreover

$$I^- = \{y \in R : y \text{ is integrally dependent on } I \text{ relative to } M\}$$

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is an ideal of R, called the *integral closure of I relative to M*, and is the largest ideal of R which has I as a reduction relative to M. In this paper, we shall indicate the dependence of I^- on the Noetherian R–module M by means of the extended notation $I^{-(M)}$.

Let R be any ring and M an R–module. For any submodule N of M, let $(N : M)$ denote the set of elements r in R such that $rM \subseteq N$. Note that $(N : M)$ is the annihilator of the module M/N, and hence $(N : M)$ is an ideal of R. A proper submodule N of M is called *prime* if whenever $rm \in N$ for some $r \in R$, $m \in N$ or $rM \subseteq N$.

1. Prime Modules

LEMMA 1.1. *Let R be any ring and M any R–module. Then a submodule N of M is prime if and only if $P = (N : M)$ is a prime ideal of R and the (R/P)–module M/N is torsion–free.* □

DEFINITION 1.2. Let R and S be commutative rings and let f be a ring homomorphism from R to S.

(1) For each ideal of S, the inverse image $f^{-1}(J)$ of J under f is an ideal of R, which is called the *contraction of J to R* and is denoted by J^c.
(2) For each ideal I of R, the ideal $f(I)S$ of S generated by $f(I)$ is called the *extension of I to S* and is denoted by I^e.

DEFINITION 1.3. We say that a subset S of a commutative ring R is *multiplicatively closed* when

(1) $1 \in S$, and
(2) If $r, s \in S$, then $rs \in S$.

Now we go back to the definition of a prime submodule and remove the submodule concept. Thus we give the definition of *prime module*.

DEFINITION 1.4. Let M be an R–module. Then M is called *prime* if whenever $rm = 0$ either $m = 0$ or $rM = 0$.

PROPOSITION 1.5. *Let M be an R–module and N a proper submodule of M. Then M/N is a prime R–module if and only if N is a prime submodule of M.*

LEMMA 1.6. *Let N and K be submodules of M. Suppose that N and K are both prime R–modules. Then $N \cap K$ is also a prime R–module.*

Note that the ring case fails.

DEFINITION 1.7. Let M be an R–module. Then we define the set

$$\text{Spec}(M) = \{L : L \text{ is a prime submodule of } M\}.$$

PROPOSITION 1.8. *Let M be a prime R–module. Let N be a non–zero submodule of M. Let $A = \{L \in \text{Spec}(M) : L \supseteq N\}$. Then $\sqrt{(0:N)} = \bigcap_{L \in A}(0:L)$.*

Proof. Take $r \in \sqrt{(0:N)}$ and $L \in A$. Then there exists a positive integer n such that $r^n N = 0$. Thus, since L is prime we get $\sqrt{(0:N)} \subseteq \bigcap_{L \in A}(0:L)$. This completes the proof since the converse is clear. □

LEMMA 1.9. *Let M be a prime R–module and K a non–zero prime R–module with $K \subseteq M$. Then $(0:M) = (0:K)$.*

COROLLARY 1.10. . *Let M be a prime R–module and let N be a non–zero submodule of M. Then $\sqrt{(0:N)} = (0:M)$.*

Proof. This is clear by Proposition 1.8 and Lemma 1.9. □

Note that if M is a prime R–module then any submodule N of M is prime.

THEOREM 1.11. *Let R be a local ring with the maximal ideal m. Let M be a finitely generated prime R–module. Then either M has a proper submodule, which is prime, or M can be expressed as a sum of simple R–modules that is, M is a semi–simple R–module.*

Proof. Let $(0:M) = P$. If P is not maximal then $P \subset m$. Then by Nakayama's lemma M has a proper submodule which is prime. If P is maximal, then consider the fact that M has a module structure over R/m. □

Now we continue to investigate prime modules.

PROPOSITION 1.12. *Let M be a prime R–module. Then any cyclic submodule of M is isomorphic to R/P for some prime ideal P of R.*

Proof. Since M is prime R–module, $P = (0 : M)$ is a prime ideal of R. For any $x \in M$, consider the isomorphism $R/(0 : x) \cong Rx$. □

Note that if M is a prime R–module, then for any non-zero x in M, $(0 : x) = (0 : M) = P$. Hence we have

COROLLARY 1.13. *Let M be a finitely generated prime R–module. Then M can be expressed as a sum of isomorphic submodules.*

PROPOSITION 1.14. *Let M be a direct sum of modules M_i, $(i \in I)$. If M is prime then M_i, $(i \in I)$ is prime.*

Note that the converse of Proposition 1.18 is not true in general as the following example illustrates.

EXAMPLE 1.15. Let $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ (as a $\mathbb{Z}$–module). Although $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are prime, M itself is not a prime $\mathbb{Z}$–module.

2. Artinian Prime Modules

In this section we continue to investigate prime modules. We now take M as an Artinian prime module over the commutative ring R. We are aiming to give the structure theorem on Artinian prime modules.

DEFINITION 2.1. An R–module M is said to be ***secondary*** if M is non–zero and, for all $r \in R$, either $rM = M$ or there exists $n \in N$ such that $r^n M = 0$.

LEMMA AND DEFINITION 2.2. *Let M be a secondary R–module. Then it may be shown that $P = \sqrt{(0 : M)}$ is a prime ideal of R, and we say that M is P–secondary.*

DEFINITION 2.3. Let M be an R-module. A secondary representation for M is an expression of the form $M = M_1 + M_2 + \ldots + M_r$, where M_i is a secondary submodule of M for all $i = 1, \ldots, r$. Note that we allow the empty sum as a secondary representation for the zero R–module. Suppose that we have a secondary representation and let $P_i = \sqrt{(0 : M_i)}$ for $i = 1, \ldots, r$. We say that the secondary representation is minimal if

(1) $P_1, \ldots, P_r$ are all distinct, and
(2) No term in the sum is redundant, that is,

$$M_i \not\subset \sum_{j \neq i}^{r} M_j, \ \forall i = 1, \ldots, r.$$

Note that any secondary representation of M can be modified to a minimal one.

UNIQUENESS THEOREM AND DEFINITION. *Suppose that the R–module M has a secondary representation, and hence a minimal secondary representation. Then the number of terms in any two minimal secondary representations for M are equal. Moreover, if $M = M_1 + M_2 + \cdots + M_r$ is a minimal secondary representation for M with $\sqrt{(0 : M_i)} = P_i$ for $i = 1, \ldots, r$, then the set $\{P_1, P_2, \ldots, P_r\}$ of prime ideals of R is independent of the choice of a minimal secondary representation for M. We denote this set by $\mathrm{Att}(M)$ and refer to its members as the **attached prime ideals** of M.*

PROPOSITION 2.5. *Let (R, m) be a local ring. Let M be an Artinian R–module. Then M is finitely generated if and only if $\mathrm{Att}(M) \subseteq \{m\}$.*

Proof. Necessity. If M is finitely generated, it is either zero or m–secondary.

Sufficiency. If $\mathrm{Att}(M) \subseteq \{m\}$, then there exists $n \in \mathbf{N}$ such that $m^n M = 0$. Hence the claim follows. □

DEFINITION 2.6. Let m be a maximal ideal of R and let L be a non-zero module over R. The m-torsion submodule $\Gamma_m(L)$ of L is defined by

$$\Gamma_m(L) = \bigcup_{n \in \mathbb{N}} (0_L : m^n).$$

THEOREM 2.7. *Let M be a non-zero Artinian prime module over a commutative ring R. Then M is finitely generated.*

Proof. By [11,(1.4)] there are only finitely many maximal ideals m of R for which $\Gamma_m(M) \neq 0$. If the distinct such maximal ideals are $m_1, \ldots, m_r$ then $M = \Gamma_{m_1}(M) \oplus \Gamma_{m_2}(M) \oplus \cdots \oplus \Gamma_{m_r}(M) = \Gamma_{m_i}(M)$ for some i, $(1 \leq i \leq r)$. Then $M_i = \Gamma_{m_i}(M_i)$, and from [11,(1.6) and (1.7)] M_i has a natural module structure over R_{m_i}. Also by [11, (1.8) and (1.13)], $\mathrm{Att}_{R_{m_i}}(M_i) = \{P_i R_{m_i} : P_i \in \mathrm{Att}_R(M_i)\}$. Then the result follows from (2.5). □

THEOREM 2.8. *Let A be non-zero Artinian prime module over a commutative ring R. Then A is Noetherian.*

Proof. By Theorem 2.7, A is finitely generated. Then $R/(0 : A) = \overline{R}$ is a Noetherian ring. This completes the proof. □

3. Integral Closure of an Ideal Relative to a Prime Module

Let R be a ring (not necessarily Noetherian). The aim of this section is to describe the integral closure of an ideal I of R relative to a Noetherian prime module M in terms of certain ideals of R.

DEFINITION AND NOTATION 3.1. Let M be a finitely generated prime R-module. Let $S = R - P$, where $P = (0 : M)$. Then S is a multiplicatively closed subset of R and each element of S is a non-zero divisior on M. Let Δ denote the set $\{K : K \text{ is an ideal of } R \text{ and } K \cap S \neq \emptyset\}$.

PROPOSITION 3.2. *Let M, P and Δ be as in (3.1). Let I be an ideal of R such that $I \subseteq P$. Then the set $\{(IKM : KM) : K \in \Delta\}$ has a maximal element.*

Proof. This clear by [13,(3.44)]. □

From now on we employ I_Δ to denote the maximal element mentioned in Proposition 3.2.

THEOREM 3.3. *Let M, P and I be as above. Then $I_\Delta = P$.*

Proof. Take $x \in I_\Delta$. Suppose M is generated by $a_1, a_2, \ldots, a_n$. Then we have $x \in (IKM : KM)$ for some $K \in \Delta$. For any $0 \neq r \in K \cap S$, we have $xrM \subseteq IKM$. Hence $xra_i = \sum_{j=1}^{t} b_{ij}a_j$ for $b_{ij} \in I$. By the standard determinant argument we can find $c_1, c_2, \ldots, c_n \in I$ such that $(-1)^n((xr)^n + c_1(xr)^{n-1} + \cdots + c_n) \in P$. Therefore $(xr)^n \in P$. Thus $x \in P$. Since the reverse inclusion is obvious we have the desired result. □

Let R be a commutative Noetherian ring. Let M be a finitely generated prime R–module. Let $P = (0 : M)$ and Δ be as in (3.1). Let I be an ideal of R with $I \not\subseteq P$. Then under these hypothesis, we have the following theorem.

THEOREM 3.4. $I_\Delta = I^{-(M)}$.

Proof. This is immediate from [14,(2.1),(2.2),(2.5) and (2.6)]. □

4. Primary Submodules

In this section we investigate the structure of primary submodules over a general commutative ring R.

LEMMA 4.1. . *Let R be a commutative ring and M be an R–module. Then N is a primary submodule of M if and only if the following are satisfied:*

(1) *If $m \in N$ then $(N : m) = R$,*
(2) *If $m \notin N$, then $\sqrt{(N : L)} = \sqrt{(N : M)}$ for all submodules L of M which contain m and N.* □

An R–module M will be called ***fully faithful*** if every non–zero submodule of M is faithful.

PROPOSITION 4.2. *A submodule N of M is primary if and only if $\sqrt{(N:M)}$ is a prime ideal of R and the R/P–module M/N is fully faithful.*

PROPOSITION 4.3. *Let R be any ring, M and M' R–modules and f be an homomorphism from M to M'. Let N be a primary submodule of M' such that $f(M) \not\subseteq N$. Then $f^{-1}(N)$ is a primary submodule of M.*

Proof. It is clear that $f^{-1}(N) \neq M$. Let $r \in R$, $x \in M$ and $rx \in f^{-1}(N)$ be such that $x \notin f^{-1}(N)$. Then $rf(x) \in N$. This completes the proof. □

The following definition is well known.

DEFINITION 4.4. Let R be a ring and let M be R–module. Let N be a submodule of M. A submodule K of M maximal with respect to the property that $N \cap K = 0$ is called a *complement of N in M*. A submodule K of M will be called a *complement in M* if there exists a submodule N of M such that K is a complement of N in M.

DEFINITION 4.5. Let R be any ring. We call a non–zero R–module M *fully compressible* if for every non–zero submodule N of M there exists a positive integer s such that M can be embedded in N^s, a direct sum of s copies of N.

PROPOSITION 4.6. *Let R be any ring. Let M be an R–module and N be a proper submodule of M.*

(1) *If M/N is a full compressible module, then N is primary submodule,*
(2) *If N is a primary submodule of M and K a submodule containing N such that K/N is a complement in M/N, then K is a primary submodule of M.*

Proof. To prove (1) we use Lemma 4.1. Let L be a submodule of M which properly contains N and take $r \in \sqrt{(N:L)}$. Then $r^k L \subseteq N$ for some k. We have a monomorphism from M/N into $(L/N)^t$, a direct sum of copies of L/N, for some positive integer t. This completes the proof.

(2) let L be a submodule of M such that $K \subset L$ and $r^k L \subseteq K$ for some $r \in R$ and a positive integer k. On the other hand, K/N is not essential in L/N so there exists a submodule T/N of L/N such that $(K/N) \cap (T/N) = 0$. Therefore $r^k x \in N$ for some $x \in L$ with $x \notin N$. Thus the result follows from Lemma 4.1 and the fact that N is primary. □

LEMMA 4.7. *Let I be an ideal of a ring R and let M be an R–module. Then there exists a proper submodule N of M such that $I = (N : M)$ if and only if $IM \neq M$ and $I = (IM : M)$.*

DEFINITION 4.8. Let R be any ring. An R–module M will be called *weakly Noetherian* if, for every element a in R and element x in M, the submodule $RaRx$ is finitely generated.

Let M be an R–module and let Q be a primary ideal of R. T hen we let $M(Q)$ denote the set

$$\{x \in M : Ax \subseteq QM \text{ for some ideal } A \not\subseteq \sqrt{Q}\}.$$

Clearly, $M(Q)$ is a submodule of M.

LEMMA 4.9. *Let Q be a primary ideal of a ring R. Let M be an R–module such that there exists a primary submodule K of M with $\sqrt{(K : M)} = \sqrt{Q}$. Then $M(Q) \subseteq K$.*

PROPOSITION 4.10. *Let Q be a primary ideal of R. Let M be a left R–module such that $R/\sqrt{Q}$–module $M/(\sqrt{Q})M$ is weakly Noetherian. Let $N = M(Q)$. Then $N = M$ or N is a primary submodule of M such that $\sqrt{Q} = \sqrt{(N : M)}$.*

Proof. Suppose that $N \neq M$. Let $r \in R$, $x \in M$ such that $rx \in N$. If $r \in \sqrt{Q}$ then there exists a positive integer k such that $r^k M \subseteq N$. Suppose that $r \notin \sqrt{Q}$. Let $A = RrR$. Then $A \not\subseteq \sqrt{Q}$. And $M/(\sqrt{Q})M$ is weakly Noetherian implies that $Ax + (\sqrt{Q})M = Rx_1 + \cdots + Rx_t + (\sqrt{Q})M$ for some positive integer t and elements $x_i \in Ax \subseteq N$. Therefore for each i, $(1 \leq i \leq t)$, there is an ideal B such that $B_i \not\subseteq \sqrt{Q}$ and $B_i x_i \subseteq QM$. Then $B = \bigcap_{i=1}^{t} B_i$ is an ideal such that $B \not\subseteq \sqrt{Q}$. Moreover $Bax \subseteq (\sqrt{Q})M$ implies $x \in N$. Thus N is a primary submodule of M.

Let $C = \sqrt{(N : M)}$. Then $\sqrt{Q} \subseteq C$. Suppose that $\sqrt{Q} \neq C$. Let $c \in C$, $c \notin \sqrt{Q}$. Let $x \in M$. Then $RcRx \subseteq N$. Then $x \in N$. This completes the proof. □

PROPOSITION 4.11. *Let R be a commutative ring and let M be a Noetherian R–module. If M/N is a uniform module then N is primary.*

Proof. Assume that $rx \in N$ but $r^k M \nsubseteq N$ for all k and $x \notin N$. Now consider the following ascending chain of submodules of M,

$$(N_M : r) \subseteq (N_M : r^2) \subseteq \cdots$$

There exists $t \in \mathbb{N}$ such that $(N_M : r^t) = (N_M : r^{t+i})$ for all $i \in \mathbb{N}$. Choose $y \in M$ satifying $r^t y \notin N$. Then $Rr^t y \nsubseteq N$, $Rx \nsubseteq N$. Let m be a non–zero element in the intersection of $(Rr^t y + N)/N$ and $(Rx + N)/N$. Thus for some s and z in R we get $sr^t y - zx \in N$. This completes the proof. □

DEFINITION 4.12. Let N be a submodule of M. Then we shall call N *irreducible* if and only if whenever $N = K \cap T$ with K, T submodules of M then $N = K$ or $N = T$.

PROPOSITION 4.13. *Let M be a Noetherian module over a commutative ring R, and let N be a proper submodule of M. If N is irreducible then N is primary.*

Proof. By [11, (9.31)] N has a decomposition. If the module in the decomposition are P-primary for some prime ideal P of R then there is nothing to prove. Assume the other case. Let $r \in R$, $m \in M$ such that $rm \in N$ with $r \notin \sqrt{(N : M)}$. Now consider the following chain of submodules of M, $(N_M : r) \subseteq (N_M : r^2) \subseteq \cdots$. Then there exists a positive integer k such that $(N_M : r^k) = (N_M : r^{k+i})$ for all $i \in \mathbb{N}$. It is clear that $N = (N + r^k M) \cap (N + mR))$. This completes the proof. □

PROPOSITION 4.14. *Let N be a primary submodule of an R–module M. Let $\{N_i\}$, $(i = 1, \ldots, n)$, be a family of submodules of M such that $\bigcap_{i=1}^n N_i \subseteq N$. If $(N_i : M) \nsubseteq \sqrt{(N : M)}$ for some i, $1 \leq i \leq n$, then $\bigcap_{j \neq i}^n N_j \subseteq N$.*

Proof. Let $n \in \bigcap_{j\neq i}^{n} N_j$. Let $r \in (N_i : M)$ such that $r \notin \sqrt{(N:M)}$. Thus $rn \in N_i$ and $rn \in \bigcap_{j=1}^{n} N_j$. This completes the proof. □

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