Honam Mathematical J. 18(1996), No. 1,pp. 1–4

SOME APPLICATION OF A NEW LEMMA

Y.K. KIM, S.H. KOH AND D.S. PARK

Dept. of Mathematics,
Chonbuk National University, Chonju, Chonbuk 560-756, Korea.

1. Introduction

Let K be a non-archimedean complete field with a nontrivial valuation $|\cdot|_v$. Suppose that a is a separable element over K and that $< b_j >$ is a sequence of separable points such that $K \subsetneq K(a) \subset K(b_j)$ and $\lim_j b_j = b$ in $(\bar{K})^c$ which is the completion of an algebraic closure $\bar{K}$ of K. Under what condition, may we then conclude that $b \in \bar{K}$ and that b is separable?

This paper sets up a lemma in §3 which is more or less a converse of the well-known Krasner's Lemma and then deals with such a problem in a proposition in §4.

2. Preliminaries

Let K be a non-archimedean complete field with a nontrivial valuation $|\cdot|_v$. We recall the well-known Krasner's Lemma without proof.

(2.1) KRASNER'S LEMMA. *Assume that a, b are given two elements of an algebraic closure $\bar{K}$ of K and that a is separable over $K(b)$. Suppose that for isomorphisms σ_i of $K(a)$ over K with $\sigma_i \neq id$, the equality $|\sigma_i(a) - a| > |b - a|$ holds. Then we have $K(b) \supset K(a)$.*

We are well aware that there is a unique extension of any valuation of K and that all conjugates of an element have the same

Received November 14. 1995.
1991, AMS Subject Classification, Primary 12J10, secondary 11S15.
Key words : non-archimedean complete field, Krasner's lemma.

valuation. The extended valuation shall also be written as $|\cdot|_v$ or simply as $|\cdot|$.

3. Presentation of a Lemma

Now we set up a sort of converse of Krasner's Lemma and prove it in an easy way. We still assume that K is as in §2.

(3.1) LEMMA. *Let $\bar{a}$ and $\bar{b}$ be separable over K with $K \subsetneq K(\bar{a}) \subset K(\bar{b})$. Then there exists $a, b \in \bar{K}$ such that $K(\bar{a}) = K(a)$, $K(\bar{b}) = K(b)$ and $|b-a|_v < |a_i - a|_v$ for all $a_i = \sigma_i(a)\,(\neq a)$ which are conjugates of a over K.*

Furthermore, we may make the distances between b and its other conjugates $\tau_j(b)$ over K with $\tau_j(a) \neq a$ no less than $\min|a - \sigma_i(a)|$ by a suitable choice of b.

Proof. Multiplying some elements in K, we may assume that there exist $\bar{\bar{a}}$ and $\bar{\bar{b}}$ so that $K(\bar{\bar{a}}) = K(\bar{a})$, $K(\bar{\bar{b}}) = K(\bar{b})$ and that the distances between $\bar{\bar{a}}$ (resp. $\bar{\bar{b}}$) and its other conjugates are bounded by any desired bounds.

If $K(\bar{a}) = K(\bar{b})$, then we are done putting $a = b = \bar{a}$. The last assertion is obvious in this case. Now suppose that $K(\bar{a}) \subsetneq K(\bar{b})$. In this case, we see by inspection of determinants that there exist infinitely many $x \in K$ such that $|x|$ is sufficiently small-in particular smaller than $\min_{i,j}\{|\bar{a} - \sigma_i(\bar{a})|_v \cdot |\bar{b}|_v^{-1},\ |\bar{a} - \tau_j(\bar{a})|_v \cdot |\bar{b} - \tau_j(\bar{b})|_v^{-1}\}$ and such that $K(\bar{a} + x\bar{b}) = K(\bar{b})$. Putting $a := \bar{a}$ and $b := \bar{a} + x\bar{b}$ finishes the proof.

We shall verify this explicitly. Letting $[K(\bar{b}) : K] = n$, we have $[K(\bar{a} + x\bar{b}) : K] \leq n$ for any $x \in K$ since $K(\bar{a}) \subset K(\bar{b})$. So $K(\bar{a} + x\bar{b}) \subset K(\bar{b})$ is obtained. For the converse containment, we shall show that $\bar{b}$ is a linear combination of $(\bar{a} + x\bar{b})^i$ with $0 \leq i \leq n-1$ for infinitely many x satisfying the above conditions. Now we make an equation

$$\bar{b} - \{k_0 + k_1(\bar{a} + x\bar{b}) + \cdots + k_i(\bar{a} + x\bar{b})^i + \cdots + k_{n-1}(\bar{a} + x\bar{b})^{n-1}\} = 0$$

with unknown k_i's.

In order to find out k_i's in K satisfying this equation, we first expand all terms and reorder the outcome to have an ascending graded expression with repect to $\bar{b}^i$ with coefficients in K, which is possible because of the assumption $\bar{a} \in K(\bar{b})$. Now making all coefficients of this equation equal to zeros, we have a non-homogeneous system of linear equations over K with n unknowns k_i. The determinants of the matrix of coefficients of this system is a polynomial in $K[x]$ of at most degree $1 + 2 + \cdots + (n-1) = n(n-1)/2$. Such a polynomial has only a finite number of roots, so that this determinant is nonzero for infinitely many $x \in K$.

In particular, we may have

$$|\bar{a} - \sigma_i(\bar{a})| = |a - \sigma_i(a)| > |x\bar{b} - \tau_j(x\bar{b})| = |x\{\bar{b} - \tau_j(\bar{b})\}|$$

at the same time for infinitely many and sufficiently small x, where τ_j's are isomorphisms of $\bar{b}$. Such a process enables us to put $b := \bar{a} + x\bar{b}$.

For, in this case, we have $K(b) = K(\bar{b})$ and $|b - a|_v = |x\bar{b}|_v = |x|_v \cdot |\bar{b}|_v < |a - \sigma_i(a)|_v \cdot |\bar{b}|_v^{-1} \cdot |\bar{b}|_v = |a - \sigma_i(a)|_v$.

Furthermore, we obtain

$$\begin{aligned} |b - \tau_j(b)|_v &= |\bar{a} + x\bar{b} - \tau_j(\bar{a} + x\bar{b})|_v \\ &= |\{\bar{a} - \tau_j(\bar{a})\} + x\{\bar{b} - \tau_j(\bar{b}\}|_v \\ &= |\bar{a} - \tau_j(\bar{a})|_v \geq \min |a - \sigma_i(a)|_v \neq 0, \end{aligned}$$

proving our lemma. □

4. Application of our Lemma

We are now prepared to establish a proposition solving the problem already posed in the introduction.

(4.1) PROPOSITION. *Let K be a non-archimedean complete field with a nontrivial valuation $|\cdot|_v$. Let a be a separable element over K and $< b_j >$ a sequence of separable points such that $K \subsetneq K(a) \subset K(b_j)$ and $\lim_j b_j = b$ in $(\bar{K})^c$. We suppose further that we may find $n \in \mathbb{Z}$ and $x \in K$ such that for all j sufficiently large*

$$\begin{aligned} 0 \neq \sup_{i,j}\{|a - \sigma_i(a)|^n \cdot |b_j - \tau_j^k(b_j)|^{-1}\} < |x| \\ < \inf_{i,j}\{|a - \sigma_i(a)| \cdot |b_j|^{-1}, |a - \tau_j^k(a)| \cdot |b_j - \tau_j^k(b_j)|^{-1}\} \end{aligned}$$

for all isomorphisms τ_j^k *of* $K(b_j)$ *over* $K(a)$. *Then* $b \in \bar{K}$ *and* b *is separable.*

Proof. By virtue of Lemma (3.1) and by the condition $|x| < \inf_{i,j}\{|a-\sigma_i(a)|\cdot|b_j|^{-1}, |a-\tau_j^k(a)|\cdot|b_j-\tau_j^k(b_j)|^{-1}\}$, there exists a sequence $< \bar{b}_j >$ with j sufficiently large such that $K(b_j) = K(\bar{b}_j)$, $|\bar{b}_j - a| < \min|a-\sigma_i(a)|$ and such that the distances between $\bar{b}_j$ and its conjugates over K obtained by isomorphisms not fixing a are not less than $\min|a-\sigma_i(a)|$.

On the other hand, $< \bar{b}_j >$ must have at least a sub-Cauchy sequence $< \bar{b}_{j_k} >$ in the set $\{y \in \bar{K} : |y-a| \leq \min|a-\sigma_i(a)|\}$. But then by the proof of Lemma (3.1) and by the condition $0 \neq \sup_{i,j}\{|a-\sigma_i(a)|^n \cdot |b_j - \tau_j^k(b_j)|^{-1}\} < |x|$, we have $|\bar{b}_{j_k} - \tau_k^l(\bar{b}_{j_k})| \geq \min_i\{\min|a-\sigma_i(a)|^n, \min|a-\sigma_i(a)|\} \neq 0$ for any isomorphisms $\tau_k^l(\neq id)$ of $K(\bar{b}_{j_k})$ over K with j_k sufficiently large. Hence by Krasner's Lemma (2.1), $K(\bar{b}_{j_k})$'s are all the same for such j_k's, from which we obtain our assertion. □

REMARK. *Here we would like to pose a problem to find out a nonexample which shows that the converse of proposition (4.1) is not true.*

References

1. S. Lang, *Algebraic Number Theory*, Springer - Verlag, 1986.
2. ——, *Algebra*, Addison - Wesley, 1984.
3. S. Warner, *Topological Fields*, North-Holland, 1989.
4. ——, *Finite extensions of valued fields*, Canad Math. Bull. 29 (1986), pp.64-69.