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CONNECTEDNESS IM KLEINEN AND LOCAL CONNECTEDNESS IN $C(X)$

JOO RAN MOON*, KUL HUR*, AND CHOON JAI RHEE**

* *Dept. of Mathematics, Wonkwang University, Iksan 570-749, Korea.*
** *Dept. of Mathematics, Wayne State University, Detroit, Michigan 48202, U.S.A.*

0. Introduction

One of the earliest results about local connectivity of hyperspace is due to Wojdyslawski. In 1939, he proved that each of 2^X and $C(X)$ is locally connected if and only if X is locally connected [16]. In 1970's Goodykoontz gave characterizations of local connectedness (connectedness im kleinen) and locally arcwise connectedness of 2^X only at singleton set $\{x\} \in 2^X$ [5, 6, 7]. In particular he proved that $C(X)$ is locally connected at $\{x\}$ if and only if $C(X)$ is connected im kleien at $\{x\}$ if and only if X is connected im kleinen at x.

In this paper, further local properties are obtained and a relationship between the sets of non-locally connected points of the space X and its hyperspace is given.

In section 1, we state a notion of property k which provides a certain structure of local property and prove that a metric continuum X has property k if and only if the projection $\bigcup\mathcal{U}$ of each open set $\mathcal{U}$ of $C(X)$ is open in X.

In section 2, we use the Hausdorff metric topology to shorten the proofs of Goodykoontz's results in [5, 6] and add several results of our own on the local connectivities of the hyperspace $C(X)$.

In section 3, we give a characterization for a point $A \in C(X)$ at which $C(X)$ is not connected im kleinen. Then we show a relationship between the set N of all points at which X is not

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connected im kleinen and the set $\mathcal{N}$ of all points at which $C(X)$ is not connected im kleinen.

1. Preliminary

Throughout the paper, X will denote a compact metric continuum with a metric d. By a continuum we mean a compact and connected space. Let 2^X be the collection of all nonempty closed subsets of X and let $C(X)$ be the collection of all subcontinua of X.

Let $A \in 2^X$ and $\epsilon > 0$. Let $N(\epsilon, A)$ be the set of all $x \in X$ such that $d(x,a) < \epsilon$ for some $a \in A$. $N(\epsilon, A)$ is called *the ϵ−neighborhood* of A. For convenience, we write $N(\epsilon, \{x\}) = N(\epsilon, x)$.

For $A, B \in 2^X$, let $H(A,B) = \inf\{\epsilon > 0 : A \subset N(\epsilon, B)$ and $B \subset N(\epsilon, A)\}$. Then H is called the ***Hausdorff metric*** for 2^X, and we call $(2^X, H)$ and $(C(X), H)$ the ***hyperspaces*** of closed sets and subcontinua respectively. Also the Hausdorff metric for 2^{2^X} is denoted by H^2.

There are two special continuous maps:

(i) [11, p.513.] $2^* : 2^X \to 2^{2^X}$ is a map defined by $2^*(A) = 2^A,\ \forall\ A \in 2^X$.

(ii) [11, p.100.] The union map $\sigma : 2^{2^X} \to 2^X$ is defined by $\sigma(\mathcal{A}) = \cup\mathcal{A},\ \forall \mathcal{A} \in 2^{2^X}$. Furthermore σ is nonexpansive, i.e, $H(\sigma\mathcal{A}, \sigma\mathcal{B}) \leq H^2(\mathcal{A}, \mathcal{B})$ for $\mathcal{A}, \mathcal{B} \in 2^{2^X}$.

In [9], Kelly introduced a notion of property 3.2, which is now called property k. The notion has been very useful in hyperspace theory. In this section we give a necessary and sufficient condition for a metric continuum to have property k.

LEMMA 1.1. *Let $A, B \in 2^X$ and $\epsilon > 0$. Then $H(A,B) < \epsilon$ if and only if $A \subset N(\epsilon, B)$ and $B \subset N(\epsilon, A)$.*

LEMMA 1.2.[11, P.34]. *If $A, B, C \in 2^X$ such that $C \subset B$, then $H(A, A \cup C) \leq H(A, B)$.*

LEMMA 1.3.[13]. *If* $A, B, C, D \in 2^X$, *then*
$H(A \cup B, C \cup D) \leq \max\{H(A,C), H(B,D)\}$.

Let D be a subset of X and let $C(D) = \{A \in C(X) : A \subset D\}$ and $2^D = \{A \in 2^X : A \subset D\}$.

LEMMA 1.4. *(a) If* $\mathcal{U}$ *is a connected subset of* 2^X *such that* $\mathcal{U} \cap C(X) \neq \emptyset$, *then* $\cup\mathcal{U}$ *is connected. In particular, if* $\mathcal{U}$ *is a connected subset of* $C(X)$, *then* $\cup\mathcal{U}$ *is connected.*
(b) A subset D *of* X *is connected if and only if* $C(D)$ *is connected.*
(b′) A subset D *of* X *is connected if and only if* 2^D *is connected.*
(c) If U *is an open subset of* X, *then* $C(U)$ *is open in* $C(X)$ *and* 2^U *is open in* 2^X.
(c′) F *is a closed subset of* X *if and only if* $C(F)$ *is closed in* $C(X)$, *and* F *is closed in* X *if and only if* 2^F *is closed in* 2^X.
(d) If P *is a component of an open subset* U *of* X, *then* $C(P)$ *is a component of* $C(U)$ *and* 2^P *is a component of* 2^U.

LEMMA 1.5. *Let* $x \in X$, *and* $\epsilon > 0$. *Let* $U = N(\epsilon, x)$ *be the* ϵ*–neighborhood of* x *in* X *and* $U^* = C(U)$. *If* $\mathcal{U}$ *be the* ϵ*–neighborhood of* $\{x\}$ *in* $C(X)$, *then* $U^* = \mathcal{U}$ *and* $U = \cup\mathcal{U}$.

REMARK. *(i) In general,* $C(\cup\mathcal{U}) \neq \mathcal{U}$. *For example, take any nondegenerate metric continuum* X. *Let* $\mathcal{U}$ *be a neighborhood of* X *in* $C(X)$ *which does not intersect* $X^* = \{\{x\} : x \in X\}$. *Then* $\cup\mathcal{U} = X$ *and* $C(X) \neq \mathcal{U}$.
(ii) If $\mathcal{U}$ *is an open set in* $C(X)$, *then* $\cup\mathcal{U}$ *may not be open in* X. *To see it, we define the space* Y *in the plane as follows : Let* $a = (0,1)$, $b_n = (1/n, 1/n)$, $c_n = (1+1/n, 0)$, $d_n = (1/n, -1/n)$, $e_n = (1/n, -1/2)$, $e = (0, -1/2)$, $p = (1,0)$ *and put* $Y_1 = ae \cup pq \cup \bigcup_{n=1}^{\infty}(ab_n \cup b_nc_n \cup c_nd_n \cup d_ne_n)$, *let* Y_2 *be the image of* Y_1 *under the symmetry with respect to the origin* $q = (0,0)$ *and finally put* $Y = Y_1 \cup Y_2$. *Let* $U = N(\frac{1}{4}, q)$, *and let* $\mathcal{V} = \{B \in C(X) : H(B, [e,e']) < \frac{1}{2}\}$. *We let* $\mathcal{U} = C(U) \cup \mathcal{V}$. *Then* $\mathcal{U}$ *is a neighborhood of* $\{q\}$. *Since* $B \subset [a,a']$ *for each* $B \in \mathcal{V}$, $\cup\mathcal{U} = U \cup \{(0,y) \in [a,a'] : \frac{-3}{4} < y < \frac{3}{4}\}$, *which is not open in* X.

DEFINITION 1.6. *Let X be a metric continuum. For $x \in X$, let $T(x) = \{A \in C(X) : x \in A\}$. $T(x)$ is called the total fiber of X at x. We say that a point $a \in X$ is a $k-$point of X provided that for each $\epsilon > 0$ there is a $\delta > 0$ if $A \in T(a)$ and b is in the $\delta-$neighborhood of a, then there is an element $B \in T(b)$ such that $H(A, B) < \epsilon$. If each point of X is a $k-$point, then we say that X has property k.*

LEMMA 1.7. *Let X be a metric continuum. If $\mathcal{U}$ is an open set in $C(X)$, then each $k-$point of X lying in $\cup\mathcal{U}$ is an interior point. On the other hand, if a point $x \in X$ has the property that whenever $\mathcal{U}$ is an open set in $C(X)$ with $\cup\mathcal{U}$ containing x $\cup\mathcal{U}$ is open in X, then x is a $k-$point of X.*

THEOREM 1.8. *A metric continuum X has property k if and only if $\cup\mathcal{U}$ is open in X for every open set $\mathcal{U}$ in $C(X)$.*

Proof. Suppose X has property k. Let $\mathcal{U}$ be an open set in $C(X)$. Then each point of $\cup\mathcal{U}$ is an interior point by Lemma 1.7. Hence $\mathcal{U}$ is open in X.

Conversely we suppose that $\cup\mathcal{U}$ is open in X for each open set $\mathcal{U}$ in $C(X)$. Let $x \in X$. We show that x is a $k-$point of X. Let $A \in C(X)$ such that $x \in A$, and let $\epsilon > 0$. Let $\mathcal{O}$ be the $\epsilon-$neighborhood of A in $C(X)$. Let $U = \cup\mathcal{O}$. Since U is open in X by assumption, there is a $\delta > 0$ such that the $\delta-$neighborhood V of x is contained in U. Let $y \in V$. Then there is an element $B \in \mathcal{O}$ such that $y \in B$. Thus x is a $k-$point of X and hence X has property k. □

THEOREM 1.9. *If the singleton set $\{x\}$ is a $k-$point of $C(X)$, then x is a $k-$point of X.*

Proof. Let $A \in T(x)$, where $T(x)$ is the total fiber of X at x. Let $\epsilon > 0$. Then $\{x\} \in C(A)$ and $C(A)$ is a subcontinuum of $C(X)$. Since $\{x\}$ is a $k-$point of $C(X)$, there exists $\delta > 0$ such that for each $B \in \mathcal{V}$, where $\mathcal{V}$ is the $\delta-$neighborhood of $\{x\}$ in $C(X)$, there is an element $\mathcal{E} \in \mathcal{T}(B)$, where $\mathcal{T}(B)$ is the total fiber of $C(X)$ at B, such that $H^2(\mathcal{E}, C(A)) < \epsilon$. Since $\cup\mathcal{V} = N(\delta, x)$, where $N(\delta, x)$ is the $\delta-$neighborhood of x in X by Lemma 1.5,

$A = \cup C(A), \cup\mathcal{E} = D$ is a subcontinuum of X, and $H(D, A) = H(\cup\mathcal{E}, \cup C(A)) \leq H^2(\mathcal{E}, C(A)) < \epsilon$. Hence x is a k-point of X. □

2. Connectedness im kleinen and local connectedness in $C(X)$

In this section, we include several results of Goodykoontz [5,6] with different proofs and added several results of our own.

DEFINITIONS 2.1. *Let $x \in X$. The space X is said to be connected (arcwise connected) im kleinen at x if for each neighborhood U of x, there is a neighborhood V of x lying in U such that if $y \in V$ then there is a connected (arcwise connected) subset of U containing both x and y. X is locally connected(locally arcwise connected) at x if for each neighborhood U of x, there is a connected (arcwise connected) neighborhood V of x lying in U.*

LEMMA 2.2. *If X is connected im kleinen at x, then x is a $k-$point of X.*

LEMMA 2.3. *Let X be a metric continuum. Let $\epsilon > 0, A \in C(X)$ and let $\mathcal{O}$ be the $\epsilon-$neighborhood of A in $C(X)$. Suppose $B \in \mathcal{O}$ (or $B \in \overline{\mathcal{O}}$) such that $A \cap B \neq \emptyset$. Then A and B can be connected by an arc in $\mathcal{O}$ (or in $\overline{\mathcal{O}}$).*

THEOREM 2.4.[1]. *Let $A \in C(X)$. Suppose A contains a point x at which X is connected im kleinen. Then $C(X)$ is arcwise connected im kleinen at A.*

Proof. Let $\epsilon > 0$. Let $\mathcal{O}_\epsilon(A)$ be the $\epsilon-$neighborhood of A. Then there is $\delta > 0$ such that $N(\delta, x) \subset N(\frac{\epsilon}{4}, x)$ such that if $y \in N(\delta, x)$ then x and y are in a connected subset C of $N(\frac{\epsilon}{4}, x)$. Let $B \in \mathcal{O}_\delta(A)$. Then $B \cap N(\delta, x) \neq \emptyset$. Let C be a connected subset of $N(\frac{\epsilon}{4}, x)$ containing x and a point $y \in B \cap N(\delta, x)$. Then the subcontinuum $A \cup B \cup \overline{C}$ is contained in $N(\frac{\epsilon}{2}, A)$. Hence there is an order arc α in $\mathcal{O}_{\frac{\epsilon}{2}}(A)$ from A to $A \cup B \cup \overline{C}$. Since $\overline{C} \subset N(\frac{\epsilon}{2}, B)$, $A \cup B \cup \overline{C} \subset N(\frac{\epsilon}{2}, B)$ so that $A \cup B \cup \overline{C} \in \mathcal{O}_{\frac{\epsilon}{2}}(B)$. Hence by Lemma 2.3 we have an order arc β in $\mathcal{O}_{\frac{\epsilon}{2}}(B)$ from B to $A \cup B \cup \overline{C}$. Since $\mathcal{O}_{\frac{\epsilon}{2}}(B) \subset \mathcal{O}_\epsilon(A)$, the connected set $\alpha \cup \beta \subset \mathcal{O}_\epsilon(A)$. □

COROLLARY 2.5. *If $A \in C(X)$ contains a point at which X is connected im kleinen, then A is a $k-$point of $C(X)$.*

Proof. Apply Theorem 2.4 and Lemma 2.2.

COROLLARY 2.6.[5]. *X is connected im kleinen at x if and only if $C(X)$ is connected im kleinen at $\{x\}$.*

Proof. If X is connected im kleinen at x then $C(X)$ is connected im kleinen at $\{x\}$ by Theorem 2.4.

Suppose $C(X)$ is connected im kleinen at $\{x\}$. Let U be a neighborhood of x in X. Let $U^* = C(U)$. Then U^* is a neighborhood of $\{x\}$ in $C(X)$. So there is an $\epsilon-$neighborhood $\mathcal{V}$ of $\{x\}$ in $C(X)$ contained in U^* such that if $B \in \mathcal{V}$ then there is a connected subset $\mathcal{C}$ of U^* containing both $\{x\}$ and B. Let $0 < \delta < \epsilon$, and let W be the $\delta-$neighborhood of x in X such that $W \subset U$. Let $y \in W$, and let $\mathcal{C}$ be a connected subset of U^* containing both $\{x\}$ and $\{y\}$. Then $C = \cup\mathcal{C}$ is connected subset of U containing both x and y. □

COROLLARY 2.6.1.[5]. *If X is locally connected at $x \in X$, then $C(X)$ is locally connected at $\{x\}$.*

Proof. Suppose X is locally connected at x, and let $\mathcal{U}$ be an ϵ - neighborhood of $\{x\}$ in $C(X)$. Then $U = \cup\mathcal{U}$ is an $\epsilon-$neighborhood of x by Lemma 1.5. Let V be a connected neighborhood of x contained in U. Then $C(U)$ is a connected neighborhood of $\{x\}$ in $C(X)$ contained in $\mathcal{U}$.

COROLLARY 2.6.2.[5]. *$C(X)$ is locally connected at $\{x\}$ if and only if X is connected im kleinen at x.*

Proof. Let $\epsilon > 0$ and $U = N(\epsilon, x)$. Then $\mathcal{U} = C(U)$ is an $\epsilon-$neighborhood of $\{x\}$ in $C(X)$. Since $C(X)$ is locally connected at $\{x\}$, there is a connected neighborhood $\mathcal{V}$ of $\{x\}$. Let $0 < \delta < \epsilon$ such that the $\delta-$neighborhood $\mathcal{W}$ of $\{x\}$ is contained in $\mathcal{V}$. Then $W = N(\delta, x) = \cup\mathcal{W}$ by Lemma 1.5. Since $\cup\mathcal{V}$ is connected by Lemma 1.4 and is contained in $\cup\mathcal{U} = U$ and $W \subset \cup\mathcal{V}, X$ is connected im kleinen at x.

Now suppose X is connected im kleinen at x. Then 2^X is connected im kleinen at $\{x\}$ by [5, Corollary 1] and hence $C(X)$

is locally arcwise connected at $\{x\}$ by [6, Theorem 1]. This implies that $C(X)$ is locally connected at $\{x\}$. □

COROLLARY 2.6.3. *If X is locally arcwise connected at $x \in X$, then $C(X)$ is locally arcwise connected at $\{x\}$.*

Proof. Let $\mathcal{U}$ be an ϵ−neighborhood of $\{x\}$ in $C(X)$ and $U = \cup\mathcal{U}$. Let V be a arcwise connected neighborhood of x in X contained in U. Let $\mathcal{V} = C(V)$ and $B \in \mathcal{V}$. Let $y \in B$ and C be an arc in V joining x and y. Then there are order arcs α in $\mathcal{V}$ from $\{x\}$ to $B \cup C$ and β from B to $B \cup C$ in $\mathcal{V}$. So that there exists an arc in $\mathcal{V}$ joining $\{x\}$ to B. □

REMARK. *The converse of Corollary 2.6.3. is not true : there is a continuum Y which is locally connected but not locally arcwise connected at a point x and $C(X)$ is locally arcwise connected at $\{x\}$. Let Y_n be the closure in the plane of the set $\{(u, v+4n) : v = sin\ (\frac{1}{u})$ for some $0 < u \leq 1\}$, and $M = \{(0, v) : v \geq -1\}$. Let Y be the one-point compactification of $\cup_{n=0}^{\infty} Y_n \cup M$. Then one sees that Y is locally connected but not locally arcwise connected at ∞. Also $C(Y)$ is locally arcwise connected at ∞.*

COROLLARY 2.6.4. *If X is arcwise connected im kleinen at x, then $C(X)$ is arcwise connected im kleinen at $\{x\}$.*

We give a different proof for the next theorem.

THEOREM 2.7.[5]. *Let $A \in C(X)$. If, for each open set U in X containing A, the component of U containing A contains A in its interior, then $C(X)$ is arcwise connected im kleinen at A.*

Proof. Let $\epsilon > 0$. Let $\mathcal{U}$ be the ϵ−neighborhood of A in $C(X)$. Let C_0 be the component of $N(\frac{\epsilon}{2}, A)$ containing A in its interior. Let $0 < \delta < \frac{\epsilon}{2}$ such that the δ−neighborhood V of a point $a \in A$ is contained in $Int(C_0)$. Let $\mathcal{V}$ be the δ−neighborhood of A in $C(X)$ and $B \in \mathcal{V}$. Then $H(A, B) < \delta$ implies that $B \cap V \neq \emptyset$ and $B \subset N(\frac{\epsilon}{2}, A)$. Hence $B \subset C_0$. Furthermore, since $A \cup B \subset C_0$, there are two order arcs α and β in $C(X)$ from A and B respectively to $\overline{C_0}$. Clearly if $D \in \alpha \cup \beta$ then $D \subset \overline{N(\frac{\epsilon}{2}, A)} \subset N(\epsilon, A)$. If $D \in \alpha$

then $A \subset D \subset N(\frac{\epsilon}{2}, D)$. So that $H(A,D) < \epsilon$. If $D \in \beta$ then $A \subset N(\delta, B)$ and $B \subset D$ imply $A \subset N(\frac{\epsilon}{2}, D)$. Hence $H(A,D) < \epsilon$. Therefore $\alpha \cup \beta \subset \mathcal{U}$ and there is an arc in $\alpha \cup \beta$ between A and B. □

The following is a generalization of Theorem 2.7.

THEOREM 2.7.1. *Let $A \in C(X)$. If, for each open set U in X containing A, the component C of U has nonempty interior such that $Int(C) \cap A \neq \emptyset$, then $C(X)$ is arcwise connected im kleinen at A.*

Proof. Let $\mathcal{U}$ be an ϵ−neighborhood of A in $C(X)$. Let C be the component of $N(\frac{\epsilon}{2}, A)$ which contains A. Let $a \in A \cap Int(C)$. Then there is $0 < \delta < \epsilon$ such that the δ−neighborhood V of a in X is contained in $Int(C)$. Let $\mathcal{V}$ be the δ−neighborhood of A in $C(X)$. Let $B \in \mathcal{V}$. Since $H(B,A) < \delta, B \cap C \neq \emptyset$ so that $B \subset C$. And $\overline{C} \subset \overline{N(\frac{\epsilon}{2}, A)}$. Hence, there are order arcs α from A to $\overline{C}$ and β from B to $\overline{C}$ in $\mathcal{U}$. Therefore $C(X)$ is arcwise connected im kleinen. □

COROLLARY 2.7.2. *Let $A \in C(X)$. If, for each open set U of X containing A, there exists a connected open set V of X containing A such that $V \subset U$, then $C(X)$ is arcwise connected im kleinen at A.*

Proof. If C is the component of U, then $V \subset C$. Hence the conclusion follows from Theorem 2.7.1. □

COROLLARY 2.8. *If $C(X)$ is locally connected at $A \in C(X)$, then it is arcwise connected im kleinen at A.*

Proof. Let $\epsilon > 0$ and let $\mathcal{U}$ be the ϵ−neighborhood of A in $C(X)$. Let $\mathcal{V}$ be a connected neighborhood of A whose closure is contained in the $\frac{\epsilon}{2}$−neighborhood of A. Let $C = \cup\overline{\mathcal{V}}$. Then $C \in C(X)$. Since $H(B,A) \leq \frac{\epsilon}{2}$ for each $B \in \overline{\mathcal{V}}, C \subset N(\epsilon, A)$. Also $A \subset N(\epsilon, C)$ so that $H(C,A) < \epsilon$. Now we have an order arc α in $\mathcal{U}$ from A to C, another one β in $\mathcal{U}$ from B to C for each $B \in \mathcal{V}$. Hence $C(X)$ is arcwise connected im kleinen. □

COROLLARY 2.9.[5]. *$C(X)$ is locally arcwise connected at $\{x\}$ if and only if $C(X)$ is connected im kleinen at $\{x\}$.*

Proof. Locally arcwise connectedness implies connected im kleinen. For the converse, we give a slightly different proof : Let $\epsilon > 0$. Let $\mathcal{U}$ be the ϵ−neighborhood of $\{x\}$ in $C(X)$. Let $0 < \delta < \epsilon$ so that the closure in $C(X)$ of the δ−neighborhood $\mathcal{V}$ is contained in $\mathcal{U}$. Since $C(X)$ is connected im kleinen at $\{x\}$, there is $0 < \tau < \delta$ such that the τ−neighborhood $\mathcal{W}$ of $\{x\}$ is contained in the component $\mathcal{C}$ of $\mathcal{V}$. Let $U = N(\epsilon, x), V = N(\delta, x)$, and $W = N(\tau, x)$. Let $U^* = C(U), V^* = C(V)$ and $W^* = C(W)$. Then by Lemma 1.5., $U^* = \mathcal{U}$ and $V^* = \mathcal{V}$ and $W^* = \mathcal{W}$. Let C be the closure in X of $\cup\mathcal{C}$. Then C is a connected subset of $\overline{V}$ and $H(\{x\}, C) \leq \delta$. Let $\mathcal{N} = \{A \in C(X) : W \cap A \neq \emptyset$ and $A \subset U\}$. Then $\{x\}, C \in \mathcal{N} \subset \mathcal{U}$. We show first that $\mathcal{N}$ is open. Let $B \in \mathcal{N}$. Since $B \subset U$, there is $\delta_1 > 0$ such that $N(\delta_1, B) \subset U$. Let $y \in W \cap B$. Then there is $\delta_2 > 0$ such that the δ_2−neighborhood V_2 of y is contained in W. We let $\pi = \min\{\delta_1, \delta_2, \tau\}$ and let $\mathcal{O}_\pi$ be the π−neighborhood of B in $C(X)$. Let $B' \in \mathcal{O}_\pi$. Then $H(B, B') < \pi$ implies that $B' \cap W \neq \emptyset$ and $B' \subset N(\pi, B) \subset N(\delta_1, B) \subset U$. So that $B' \in \mathcal{N}$. Hence $\mathcal{O}_\pi \subset \mathcal{N}$. Now we show that $\mathcal{N}$ is arcwise connected. Let $B_i \in \mathcal{N}, i = 1, 2$. Then $C \cap B_i \neq \emptyset$ for each i and $C \cup B_1 \cup B_2 \subset U$ that $C \cup B_1 \cup B_2 \in \mathcal{N}$. Let α_i be order arc in $C(X)$ from B_i to $C \cup B_1 \cup B_2$ for $i = 1, 2$. It is easy to see that $\alpha \cup \beta \subset \mathcal{N}$. Hence there is an arc in $\mathcal{N}$ between B_1 and B_2. This completes the proof. □

3. Decomposition of set of points of non-connected im kleinen

Let X be a metric continuum. Let N be the set of all $x \in X$ at which X is not connected im kleinen, and let $\mathcal{N}$ be the set of all $A \in C(X)$ at which $C(X)$ is not connected im kleinen, let $\mathcal{K}$ be the set of all $A \in C(X)$ at which $C(X)$ is connected im kleinen, and finally let $\mathcal{L}$ be the set of all $A \in C(N)$ at which $C(X)$ is connected im kleinen. We may note here that if $N \neq \emptyset$ then each of the components of N is nondegenerate [12,5.13]. Let us note that $C(X) = \mathcal{N} \cup \mathcal{K}$ and $\mathcal{L} \subset \mathcal{K}$.

In this section, we give a necessary and sufficient condition for which $C(X)$ is not connected im kleinen (Theorem3.3). Then we show that there is a relation between N and $\mathcal{N}$.

THEOREM 3.1.[16]. *Let X be a metric continuum. The followings are equivalent :*
(1) X is locally connected. (2) $\mathcal{N} = \emptyset$. (3) $C(X)$ is locally connected. (4) $N = \emptyset$.

Proof. If a continuum X is connected im kleinen at each of its points, then it is locally connected.
(1) ⇒ (2) : Let $A \in C(X)$ and let U be an open set in X containing A. Then by local connectedness of X, the component of U containing A is open. Hence by Theorem 2.7 $C(X)$ is arcwise connected im kleinen at A. Therefore $C(X)$ is locally connected, i.e., $\mathcal{N} = \emptyset$.
(2) ⇒ (3) : Obvious. **(3) ⇒ (4) :** $N \neq \emptyset \Rightarrow \mathcal{N} \neq \emptyset \Rightarrow C(X)$ is not locally connected. **(4) ⇒ (1) :** Obvious. □

PROPOSITION 3.2. *Suppose $C(X)$ is not connected im kleinen at $A \in C(X)$ (i.e., $A \in C(X) \backslash \mathcal{K}$). Then $A \subset N$. If $A \in \overline{\mathcal{N}}$ then $A \in C(\overline{N})$.*

Proof. The first part is a consequence of Theorem 2.4. If $A \in \overline{\mathcal{N}}$ then there is a sequence $\{A_n\}$ in $\mathcal{N}$ which converges to A. Hence $A \subset \overline{N}$. □

THEOREM 3.3. *$C(X)$ is not connected im kleinen at $A \in C(X)$ if and only if (1) $A \in C(N)$ and (2) there is an open set U containing A having a sequence $\{C_n\}$ of components and a sequence $\{A_n\}$ of subcontinua of X with $A_n \subset C_n$ for each n which converges to A.*

Proof. Suppose $C(X)$ is not connected im kleinen at A. Then by Proposition 3.2, $A \in C(N)$, and there exists an ϵ−neighborhood $\mathcal{U}$ of A in $C(X)$ and a sequences $\{\mathcal{C}_n\}$ of distinct components of $\overline{\mathcal{U}}$ which converges to a limit continuum $\mathcal{C}$ which contains A [15, Theorem 12.1, p.18], i.e., $Lt\mathcal{C}_n = \mathcal{C}$. Therefore there is a sequence

$\{A_n\}$ of subcontinua of $X, A_n \in \mathcal{C}_n$, such that $\lim A_n = A$. Let p be a positive integer such that $H(A_n, A) < \frac{\epsilon}{10}$ for all $n > p$ and $U = N(\frac{\epsilon}{10}, A)$. Then $A_n \subset U$ for all $n > p$. Let $\overline{C}$ be the closure of the component C of U containing A, and for each positive integer $n > p$ let $\overline{C}_n$ be the closure of the component C_n of U containing A_n. We claim that $\overline{C} \cap A_n = \emptyset$ for each $n > p$ and $A_n \cap A_m = \emptyset$ for $m \neq n$ and $n, m > p$. First we note that $H(\overline{C}, A) < \frac{\epsilon}{9}$ and $H(\overline{C}_n, A) < \frac{\epsilon}{9}$ for each $n > p$. Hence $H(\overline{C}_n, \overline{C}_m) < \frac{2\epsilon}{9}$ for $n, m > p$. Now suppose that $\overline{C} \cap A_n \neq \emptyset$ for some $n > p$. Then, since $A_n \subset N(\frac{\epsilon}{10}, A)$ implies $A_n \cap (\overline{C} \backslash C) = \emptyset, A_n \cap C \neq \emptyset$. Hence $A_n \subset C \subset \overline{C}$. Let α be an order arc in $C(X)$ from A_n to $\overline{C}$ and let $B \in \alpha$. Then, since $A \subset B \subset \overline{C} \subset N(\frac{\epsilon}{9}, A)$ and $A \subset N(\frac{\epsilon}{9}, B), H(B, A) < \frac{\epsilon}{9} < \epsilon$. Hence $B \in \mathcal{U}$. Thus α is an order arc in $\mathcal{U}$ from A_n to $\overline{C}$. There is also an order arc in $\mathcal{U}$ from A to $\overline{C}$ by Lemma 2.3. This means that A and A_n are in the same component of $\mathcal{U}$ which contradicts the choice of A_n. Now we show that $\overline{C}_n \cap \overline{C}_m = \emptyset$ for $m \neq n$. Suppose $\overline{C}_n \cap \overline{C}_m \neq \emptyset$. Then $\overline{C}_n \cup \overline{C}_m$ is a subcontinuum of X and $H(\overline{C}_n \cup \overline{C}_m, A) = H(\overline{C}_n \cup \overline{C}_m, A \cup A) \leq H(\overline{C}_n, A) + H(\overline{C}_m, A) < \frac{2\epsilon}{9}$ by Lemma 1.3 so that $\overline{C}_n \cup \overline{C}_m \in \mathcal{U}$. Let α be an order arc in $C(X)$ from A_n to $\overline{C}_n \cup \overline{C}_m$ and $B \in \alpha$. Then $H(B, A) < \frac{2\epsilon}{9}$ so that $B \in \mathcal{U}$. Hence α is an order arc in $\mathcal{U}$. Similarly, there is an order arc in $\mathcal{U}$ from A_m to $\overline{C}_n \cup \overline{C}_m$. Thus A_n and A_m are in the same component of $\overline{\mathcal{U}}$, which contradicts the choice of A_n and A_m. So we must have $\overline{C}_n \cap \overline{C}_m = \emptyset$. Since A_n and A_m are contained in $\overline{C}_n$ and $\overline{C}_m$ respectively, $A_n \cap A_m = \emptyset$.

For the converse, we have $C(U)$ is open in $C(X)$ and $\{C(C_n)\}$ is a sequence of distinct components of $C(U)$ by Lemma 1.4. If $\mathcal{V}$ is any δ−neighborhood of A contained in $C(U)$, then there is a positive integer p such that $A_n \in \mathcal{V}$ for all $n > p$. This means that there is no connected subset of $C(U)$ containing both A and A_n for $n > p$. Hence $C(X)$ is not connected im kleinen at A. □

COROLLARY 3.4. *If* $N \neq \emptyset$, *then* $C(N) = \mathcal{N} \cup \mathcal{L}$ *and* $\mathcal{L} \cap \mathcal{N} = \emptyset$.

Proof. If $x \in N$ then $\{x\} \notin \mathcal{K}$ by Corollary 2.6. Hence $\{x\} \in \mathcal{N}$. If A is a nondegenerate subcontinuum contained in N, then either $A \in \mathcal{L}$ or $A \in \mathcal{N}$. □

COROLLARY 3.5. *Let $\mathcal{N}_f$ be a component of $\mathcal{N}$. Then $\cup\mathcal{N}_f \subset N_f$, where N_f is a component of N.*

Proof. Since $\mathcal{N}_f$ is connected and each elememt of it is contained in a component of N, and $\cup\mathcal{N}_f$ is connected by Lemma 1.1, $\cup\mathcal{N}_f$ must be contained in a component of N. □

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