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EXACT SEQUENCES OF pR–HOMOMORHPISMS

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Throughout the following, R will denote a ring with unit element 1 and R-mod will denote the category of all unitary left R-modules.

Following the definition in [4], we say that a *fuzzy left R-module* is a pair (M, ϕ_M) consisting of a left R-module M and a function ϕ_M from M to $[0,1]$ satisfying the following conditions:

(1) $\phi_M(0_M) = 1$,
(2) $\phi_M(m+m') \geq \min\{\phi_M(m), \phi_M(m')\}$ for all $m, m' \in M$,
(3) $\phi_M(am) \geq \phi_M(m)$ for all $m \in M$ and all $a \in R$.

The notion of a fuzzy module is closely tied to the notion of a normed module studied by Fleischer in [1]. Changing his definition slightly, by a *pseudonormed left R-module* [2] with values in [0, 1] we mean a pair (M, α) consisting of a left R-module M and a function $\alpha : M \to [0,1]$ satisfying the following conditions:

(4) $\alpha(0_M) = 0$,
(5) $\alpha(m+m') \leq \max\{\alpha(m), \alpha(m')\}$,
(6) $\alpha(rm) \leq \alpha(m)$,

for all $m, m' \in M$ and $r \in R$. We denote it by α_M. Note from (6) that $\alpha(-m) = \alpha(m)$ for all $m \in M$.

In what follows, a pseudonormed R-module (briefly, pR-module) will mean a pseudonormed left R-module.

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DEFINITION 1. Let α_M and β_N be pR-modules. A function $\widetilde{f} : \alpha_M \to \beta_N$ is called a *pseudonormed R-homomorphism* (briefly, *pR-homomorphism*) if

(7) $f : M \to N$ is an R-homomorphism.
(8) $\alpha(m) \geq \beta(f(m))$ for all $m \in M$.

We call $\beta_{\text{im}f}$ the *image* of $\widetilde{f}$, denoted by $\text{im}\widetilde{f}$, and α_{M_0} the *kernel* of $\widetilde{f}$, denoted by $\ker\widetilde{f}$, where $M_0 = \{m \in M | \beta(f(m)) = 0\}$.

We note that $\ker f \subseteq \ker\widetilde{f}$. For any $m \in \ker f$, $\beta(f(m)) = \beta(0_N) = 0$ and so $m \in \ker\widetilde{f}$. As $\text{im}f$ is a submodule of N, $\text{im}\widetilde{f}$ is a pseudonormed submodule of β_N. Let $m, m' \in \ker\widetilde{f}$ and $r, r' \in R$. Then

$$\begin{aligned}\beta(f(rm + r'm')) &= \beta(f(rm) + f(r'm')) \\ &= \beta(rf(m) + r'f(m')) \\ &\leq \max\{\beta(rf(m)), \beta(r'f(m'))\} \\ &\leq \max\{\beta(f(m)), \beta(f(m'))\} \\ &= 0.\end{aligned}$$

Hence $rm + r'm' \in \ker\widetilde{f}$ and so $\ker\widetilde{f}$ is a pseudonormedsubmodule of α_M.

DEFINITION 2. Let $\widetilde{f} : \alpha_M \to \beta_N$ be a pR-homomorphism. We call $\widetilde{f}$ *epic* (resp. *monic)* if f is epic (resp. monic). We say that $\widetilde{f}$ is *quasi-monic* if $\ker\widetilde{f} = \alpha_{M'}$, where $M' = \{m \in M | \alpha(m) = 0\}$.

REMARK 3. It is clear that if $\ker\widetilde{f} = \{0\}$, then quasi-monic is just ordinary monic.

DEFINITION 4. A sequence of pR-homomorphisms

$$\alpha_M \xrightarrow{\widetilde{f}} \beta_N \xrightarrow{\widetilde{g}} \gamma_P$$

is *exact* at β_N if $\text{im}\widetilde{f} = \ker\widetilde{g}$.

A sequence of pR-homomorphisms

$$\cdots \xrightarrow{\widetilde{f_{i-1}}} \alpha^{(i-1)}_{M_{i-1}} \xrightarrow{\widetilde{f_i}} \alpha^{(i)}_{M_i} \xrightarrow{\widetilde{f_{i+1}}} \alpha^{(i+1)}_{M_{i+1}} \xrightarrow{\widetilde{f_{i+2}}} \cdots$$

is *exact* if each adjacent pair of pR-homomorphisms is exact.

The exact sequence of pR-homomorphisms

$$\delta_0 \xrightarrow{\widetilde{i}} \alpha_M \xrightarrow{\widetilde{f}} \beta_N \xrightarrow{\widetilde{g}} \gamma_P \xrightarrow{\widetilde{j}} \delta_0$$

is called the *short exact sequence* of pR-homomorphisms, where δ_0 is a singular pR-module, that is, $\delta(m) = 0$ for all $m \in M$.

Let α_A be any pseudonormed submodule of a pR-module α_M. Then a map $\bar{\alpha} : M/A \to [0,1]$ given by

$$\bar{\alpha}(m + A) = \begin{cases} 0 & \text{if } m \in A \\ \sup\{\alpha(k) | k \in m + A\} & \text{if } m \notin A \end{cases}$$

can determined the pseudonormed quotient R-module, denoted by $\bar{\alpha}_{M/A}$.

PROPOSITION 5. *If α_M is a pR-module and if m and m' are elements of M satisfying $\alpha(m) \neq \alpha(m')$, then $\alpha(m+m') = \max\{\alpha(m), \alpha(m')\}$.*

Proof. Without loss of generality, we may assume that $\alpha(m) > \alpha(m')$. Then by (5), $\alpha(m+m') \leq \alpha(m)$. Now

$$\begin{aligned} \alpha(m) &= \alpha(m + m' - m') \\ &\leq \max\{\alpha(m+m'), \alpha(-m')\} \\ &= \max\{\alpha(m+m'), \alpha(m')\} \\ &= \alpha(m+m') \end{aligned}$$

because $\alpha(m) > \alpha(m')$. Hence $\alpha(m) = \alpha(m+m')$ and therefore $\alpha(m+m') = \max\{\alpha(m), \alpha(m')\}$. □

THEOREM 6. *Let*

$$\delta_0 \xrightarrow{\widetilde{i}} \alpha_M \xrightarrow{\widetilde{f}} \beta_N \xrightarrow{\widetilde{g}} \gamma_P \xrightarrow{\widetilde{j}} \delta_0$$

be a short exact sequence of pR-homomorphisms. Then

(a) $\mathrm{im}\widetilde{i} = \ker\widetilde{f} = \alpha_{M'}$,

(b) $\mathrm{im}\widetilde{f} = \ker\widetilde{g} \supseteq \beta_{N'}$,

(c) $\widetilde{g}$ *is epic,*

where $M' = \{m \in M | \alpha(m) = 0\}$ *and* $N' = \{n \in N | \beta(n) = 0\}$.

Proof. (a): Assume that $\mathrm{im}\widetilde{i}=\ker\widetilde{f} \neq \alpha_{M'}$. If $\ker\widetilde{f} \not\subseteq \alpha_{M'}$, then there is some $m \in \ker\widetilde{f}$ such that $m \notin \alpha_{M'}$. Hence $m \in \mathrm{im}\widetilde{i}$ and so $\widetilde{i}(c) = m$ for some $c \in \delta_0$. Since $\widetilde{i}$ is a pR-homomorphism, therefore $\alpha(m) = \alpha(\widetilde{i}(c)) \leq \delta(c) = 0$, which implies that $\alpha(m) = 0$. Hence $m \in \alpha_{M'}$, a contradiction.

If $\ker\widetilde{f} \not\supseteq \alpha_{M'}$, then there exists $m \in \alpha_{M'}$ such that $m \notin \ker\widetilde{f}$. Thus $\alpha(m) = 0$, and so $\beta(\widetilde{f}(m)) \leq \alpha(m) = 0$ which implies that $\beta(\widetilde{f}(m)) = 0$. Hence $m \in \ker\widetilde{f}$. This is a contradiction. Consequently we have the result (a).

(b): Since $\widetilde{g}$ is a pR-homomorphism, we have that for any $n \in \beta_{N'}$, $\gamma(\widetilde{g}(n)) \leq \beta(n) = 0$, and that $\gamma(\widetilde{g}(n)) = 0$. This implies that $n \in \ker\widetilde{g}$.

(c): For any $x \in \gamma_P$, we have that $\delta(\widetilde{j}(x)) = 0$. Thus $x \in \ker\widetilde{j}$, and so $\ker\widetilde{j} = \gamma_P = \mathrm{im}\widetilde{g}$. Therefore $\widetilde{g}$ is epic. □

DEFINITION 7. Let $\widetilde{f} : \alpha_M \to \beta_N$ be a pR-homomorphism. Then $\widetilde{f}$ is called a ***pseudonormed weak isomorphism***, denoted $\alpha_M \cong_w \beta_N$, if $f : M \to N$ is an R-isomorphism.

THEOREM 8. *Let*

$$\delta_0 \xrightarrow{\widetilde{i}} \alpha_M \xrightarrow{\widetilde{f}} \beta_N \xrightarrow{\widetilde{g}} \gamma_P \xrightarrow{\widetilde{j}} \delta_0$$

be a short exact sequence of pR-homomorphisms. Then we have

$$\bar{\alpha}_{M/ker\widetilde{f}} \cong_w \bar{\beta}_{im\widetilde{f}/N'} \text{ and } \bar{\beta}_{N/ker\widetilde{g}} \cong_w \bar{\gamma}_{im\widetilde{g}/P'}$$

where $N' = \{a \in N | \beta(a) = 0\}$ *and* $P' = \{x \in P | \gamma(x) = 0\}$.

Proof. Define a map $\varepsilon : M \to im\widetilde{f}/N'$ by $\varepsilon(m) = \widetilde{f}(m) + N'$. For any $m_1, m_2 \in M$ and $r_1, r_2 \in R$,

$$\begin{aligned}\varepsilon(r_1m_1 + r_2m_2) &= \widetilde{f}(r_1m_1 + r_2m_2) + N' \\ &= (r_1\widetilde{f}(m_1) + r_2\widetilde{f}(m_2)) + N' \\ &= (r_1\widetilde{f}(m_1) + N') + (r_2\widetilde{f}(m_2) + N') \\ &= r_1\varepsilon(m_1) + r_2\varepsilon(m_2).\end{aligned}$$

Thus ε is an R-homomorphism. Since im$\widetilde{f}$=ker$\widetilde{g}$, therefore ε is an epic and the derived map $\varepsilon_0 : M/ker\varepsilon \to im\widetilde{f}/N'$ is an isomorphism. Note that

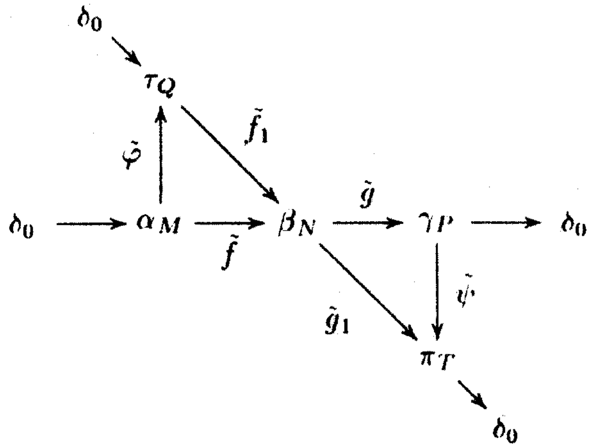
$$\begin{aligned}ker\varepsilon &= \{m \in M | \varepsilon(m) \in N'\} \\ &= \{m \in M | \widetilde{f}(m) + N' \subset N'\} \\ &= \{m \in M | \widetilde{f}(m) \in N'\} \\ &= \{m \in M | \beta(\widetilde{f}(m)) = 0\} \\ &= ker\widetilde{f}.\end{aligned}$$

Hence $M/ker\widetilde{f} \cong im\widetilde{f}/N'$. Let $m + ker\widetilde{f} \in M/ker\widetilde{f}$. If $m \in ker\widetilde{f}$, then $\bar{\alpha}(m + ker\widetilde{f}) = 0$. By the definition of $ker\widetilde{f}$, we have $\beta(\widetilde{f}(m)) = 0$ and so $\widetilde{f}(m) \in N'$. Thus $\bar{\beta}(\widetilde{f}(m) + N') = 0$. If $m \in M - ker\widetilde{f}$, then $\beta(\widetilde{f}(m)) > 0$ and $\widetilde{f}(m) \notin N'$. For every $a' \in im\widetilde{f}$, there exists $m' \in M$ such that $\widetilde{f}(m') = a'$. Thus

$$\begin{aligned}\bar{\beta}(\widetilde{f}(m) + N') &= \sup\{\beta(a') | a' \in \widetilde{f}(m) + N'\} \\ &= \sup\{\beta(f(m')) | \widetilde{f}(m') \in \widetilde{f}(m) + N'\} \\ &= \sup\{\beta(f(m')) | \widetilde{f}(m' - m) \in N'\} \\ &= \sup\{\beta(f(m')) | m' - m \in ker\widetilde{f}\} \\ &= \sup\{\alpha(m') | m' \in m + ker\widetilde{f}\} \\ &= \bar{\alpha}(m + ker\widetilde{f}).\end{aligned}$$

Hence $\bar{\beta}(\tilde{f}(m)+N') \le \bar{\alpha}(m+ker\tilde{f})$ for any $m+ker\tilde{f} \in M/ker\tilde{f}$. Consequently, we have $\bar{\alpha}_{M/ker\tilde{f}} \cong_w \bar{\beta}_{im\tilde{f}/N'}$. By the same manner, we get $\bar{\beta}_{N/ker\tilde{g}} \cong_w \bar{\gamma}_{im\tilde{g}/P'}$. □

THEOREM 9. *Consider the commutative diagram of two exact sequences of pR-homomorphisms*



(a) *$\tilde{\varphi}$ is quasi-monic and $\tilde{\psi}$ is epic.*
(b) *If $\tilde{\varphi}$ is epic, then $\tilde{\psi}$ is quasi-monic.*

Proof. (a): Let $m \in ker\tilde{\varphi}$. Then $\tau(\tilde{\varphi}(m)) = 0$ and so $\tilde{\varphi}(m) \in \tau_{Q'}$, where $Q' = \{q \in Q | \tau(q) = 0\}$. Since $\beta(\tilde{f}_1(\tilde{\varphi}(m))) \le \tau(\tilde{\varphi}(m)) = 0$, we have $\tilde{f}_1\tilde{\varphi}(m) \in \beta_{N'}$, where $N' = \{n \in N | \beta(n) = 0\}$. Using the commutativity of the diagram, then $\tilde{f}(m) = \tilde{f}_1\tilde{\varphi}(m) \in \beta_{N'}$. Thus $m \in ker\tilde{f} = \alpha_{M'}$, where $M' = \{m \in M | \alpha(m) = 0\}$. Therefore $ker\tilde{\varphi} = \alpha_{M'}$, i.e., $\tilde{\varphi}$ is quasi-monic. As $\tilde{g}_1$ is epic, ψ is also epic by commutativity of the diagram, and so $\tilde{\psi}$ is epic.

(b): Assume that $\tilde{\varphi}$ is epic. Let $p \in ker\tilde{\psi}$. Then $\pi(\tilde{\psi}(p)) = 0$ and so $\tilde{\psi}(p) \in \pi_{T'}$, where $T' = \{t \in T | \pi(t) = 0\}$. Since $\tilde{g}$ is epic, there exists $n \in N$ such that $\tilde{g}(n) = p$. It follows from the commutativity of the diagram that $\tilde{g}_1(n) = \tilde{\psi}\tilde{g}(n) = \tilde{\psi}(p) \in \pi_{T'}$. Thus $n \in ker\tilde{g}_1 = im\tilde{f}_1$ and so there is an element $q \in Q$ with

$\tilde{f}_1(q) = n$. Since $\tilde{\varphi}$ is epic, we have $\tilde{\varphi}(m) = q$ for some $m \in M$. By commutativity of the diagram we get $\tilde{f}(m) = \tilde{f}_1\tilde{\varphi}(m) = \tilde{f}_1(q) = n$. It follows that $n \in im\tilde{f} = ker\tilde{g}$, so that $\gamma(\tilde{g}\tilde{f}(m)) = \gamma(\tilde{g}(n)) = \gamma(p) = 0$. Thus $p \in \gamma_{P'}$, where $P' = \{p \in P | \gamma(p) = 0\}$. We finally obtain $ker\tilde{\psi} = \gamma_{P'}$ and $\tilde{\psi}$ is quasi-monic. □

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