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A CONDITION FOR A COMPACT EINSTEIN SPACE TO BE A SPHERE

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1. Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold with the Riemannian connection ∇. For a smooth function f on M^n, the Hessian H^f of f is a symmetric (0,2) tensor field on M^n defined by

$$(1.1) \qquad H^f(X,Y) = <\nabla_X gradf, Y>, \quad X,Y \in TM.$$

If f is a constant function, then it is obvious that H^f is proportional to the metric tensor g. And the round sphere $S^n(r)$ admits a nonconstant function of which Hessian is proportional to the metric tensor (see § 2).

Hence it is natural to ask the following question :

On which Riemannian manifolds, does there exist a nonconstant function whose Hessian is proportional to the metric tensor ?

In this article, we prove the following :

THEOREM 1. *Let (M^n, g) be a compact Einstein manifold with constant scalar curvature. Then there exists a nonconstant function such that H^f is proportional to g if and only if M^n is a round sphere $S^n(r)$.*

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THEOREM 2. *Let (M^n, g) be a compact Riemannian manifold with negative semi-definite Ricci tensor. Then the Hessian of a smooth function f on M^n is proportional to the metric tensor if and only if f is a constant function.*

Note that H^f is proportional to g if and only if f satisfies

$$\nabla_X gradf = \frac{\Delta f}{n} X, \quad X \in TM, \tag{1.2}$$

where Δf is the Laplacian of f defined by $\Delta f = tr(H^f)$.

2. Examples

Let M^n be an n-dimensional Euclidean space R^n. If f is a function of the form

$$f(x_1, \cdots, x_n) = a \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} b_i x_i + c,$$

then the Hessisn of f is proportional to the metric tensor. For a round sphere $S^n(r)$ in R^{n+1}, if we define

$$f(x_1, \cdots, x_{n+1}) = \sum_{i=1}^{n+1} a_i x_i + b,$$

then the Hessian of $f|_{s^n(r)}$ is proportional to the metric tensor.

In general, let (M^n, g) be the warped product space $R \times_w F^{n-1}$ with metric tensor $g = dt^2 + w^2(t) g_F$ ([4]). Then for the function f defined by

$$f(t) = a \int_0^t w(t) dt + b, \quad a, b \in R,$$

it can be shown that H^f is proportional to g.

3. Proofs

Let (M^n, g) be a compact Einstein manifold with constant scalar curvature. Then we have

$$Ric(X,Y) = k(n-1)g(X,Y), \tag{3.1}$$

where k is a constant.

If the Hessian of a function f is proportional to g, then from (1.2) we obtain the following :

$$R(X,Y)\nabla f = X(\frac{\Delta f}{n})Y - Y(\frac{\Delta f}{n})X, \tag{3.2}$$

$$Ric(X,\nabla f) = -(n-1)X(\frac{\Delta f}{n}). \tag{3.3}$$

where ∇f denotes the gradient vector field of f.

(3.1) and (3.3) imply that

$$X(\frac{\Delta f}{n}) + kX(f) = 0, \quad X \in TM.$$

Hence we see that

$$\Delta f + nkf = c, \tag{3.4}$$

where c is a constant.

Suppose that f is a nonconstant function. Then (3.4) implies that k is a nonzero constant and $h = f - \frac{c}{nk}$ is a nonconstant eigenfunction of (M^n, g) with eigenvalue $nk > 0$. Therefore the theorem of Obata ([3]) completes the proof of Theorem 1.

As a corollary of Theorem 1, we prove the following :

COROLLARY. *Let M^2 be a compact 2-dimensional Riemannian manifold. Then there is a nonconstant eigenfunction of M^2 whose*

Hessian is proportional to the metric tensor if and only if M^2 is a round sphere $S^2(r)$.

proof. Suppose that a nonconstant function f on M^2 satisfies (1.2) and $\Delta f + \lambda f = 0, \quad \lambda > 0$. Then we have from (3.2)

$$R(X,Y)\nabla f = \frac{\lambda}{2}[< \nabla f, Y > X - < \nabla f, X > Y]. \tag{3.5}$$

Let $U = \{p \in M^2 | \nabla f(p) \neq 0\}$. Then (3.5) implies that the Gaussian curvature K is a constant $\frac{\lambda}{2}$ on U. Suppose that $V = int(M^2 \setminus U)$ is nonempty. Then the eigenfunction f vanishes on the open set V. This contradiction shows that V is empty, hence we have $K = \frac{\lambda}{2}$ on M^2.
Thus the corollary follows. *Q.E.D.*

To prove Theorem 2, we consider the Bochner - Lichnerowicz formula ([1, p.131]) :

$$\frac{1}{2}\Delta(|\nabla f|^2) = |H^f|^2 + < \nabla f, \nabla(\Delta f) > + Ric(\nabla f, \nabla f). \tag{3.6}$$

If $f \in C^\infty(M)$ satisfies (1.2), then we have

$$|H^f|^2 = \frac{(\Delta f)^2}{n}, \tag{3.7}$$

$$Ric(\nabla f, \nabla f) = -\frac{n-1}{n} < \nabla f, \nabla(\Delta f) > . \tag{3.8}$$

Thus (3.6), (3.7) and (3.8) combine to imply

$$\frac{1}{2}\Delta(|\nabla f|^2) = \frac{1}{n}[(\Delta f)^2 + < \nabla f, \nabla(\Delta f) >] \tag{3.9}$$

on all of M.

Suppose that the Ricci tensor of M^n is negative semidefinite. Then (3.8) implies that each summand of the righthand side is nonnegative. Integrating (3.9) over M^n, we see that $\Delta f = 0$ on M^n. Hence f is a constant function. This completes the proof of Theorem 2.

References

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