JOURNAL OF THE KOREAN ASTRONOMICAL SOCIETY
29: S285 ~ S289, 1996

NON-GEOMETRICAL CONSTANTS OF THE MOTION FOR THE MAXIMALLY SYMMETRIC SPACETIMES

M. H. DEHGHANI
Physics Department and Biruni Observatory, Shiraz University, Shiraz 71454, Iran

ABSTRACT

A systematric method of exploring the *"geometrical"* and *"non-geometrical"* constants of the motion for an arbitrary spacetime is presented. This is done by introducing a series of coupled differential equation for the generators of the symmetry group of Vlasov's equation. The method is applied to the case of the maximaly symmetric spectime, and the geometrical and non-geometrical constants of motion are obtained.

The geometrical constants of the motion of an spacetime are

$$Y_{(a)} = g_{\mu\nu}\xi^{\mu}_{(a)}p^{\mu}, \tag{1}$$

where p^{μ}'s are the component of four momentum and $\xi^{\mu}_{(a)}$ are Killing's vector which can be obtained by solving Killing's equation

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \tag{2}$$

The geometrical constants of the motion of Eq. (1) are due to the geometrical symmetries of the spacetime. But, an spacetime may have more constants than the geometrical ones (Maartans & Maharaj, 1987). One may draw a parallel between this and the case of Coulomb or harmonic potential. As it is known the Coulomb potential has six constants of motion. Three of them are the components of angular momentum which are due to the geometrical symmetries of the Coulomb potential. The other three constants which are known as the dynamical constants are the components of Runge-Lenz vector. These latters are due to the symmetries of Hamiltonian or classical Liouville's operator in the six dimensional phase space.

In this paper we give a systematic method of exploring the geometrical and non-geometrical constants of the motion of an arbitrary spacetime. To do this we first study the symmetries of Vlasov's equation in the seven dimensional phase space. Vlasov's (Liouville's) equation may be written as

$$\mathcal{L}f = (p^{\mu}\frac{\partial}{\partial x^{\mu}} - \Gamma^{i}_{\mu\nu}p^{\mu}p^{\nu}\frac{\partial}{\partial p^{i}})f(x^{\mu}, p^{i}) = 0, \tag{3}$$

where (x^{μ}, p^{i}) denote the collection of configuration and momentum coordinates in the seven dimensional phase space $P(m)$, $f(x^{\mu}, p^{i})$ is a one particle distribution function, $\mathcal{L}$ is Liouville's operator and $\Gamma^{i}_{\mu\nu}$'s are Christoffel symbols. See for more details Ehler (1971). Through out this paper Greek indices run from 0 to 3, and Latin indices run from 1 to 3.

For a systematic study of the symmetries of Eq. (1) we follow a procedure parallel to that of Dehghani & Sobouti (1992, 1993, 1995). We look for those infinitesimal transformation of (x^{μ}, p^{i}) to (x'^{μ}, p'^{i}) that leave $\mathcal{L}$ form invariant. Hence let

$$x'^{\mu} = x^{\mu} + \epsilon\xi^{\mu}(x^{\mu}, p^{i}), \tag{4}$$

$$p'^{i} = p^{i} + \epsilon\eta^{i}(x^{\mu}, p^{i}), \tag{5}$$

where ϵ is an infinitesimal parameter and ξ^{μ} and η^{i} are arbitrary functions of x^{μ} and p^{i}. It is a matter of straight-forward calculation to show that the form invariance of $\mathcal{L}$ leads to the following differential equations for ξ^{μ} and η^{i}

$$\mathcal{L}\xi^{o} = \chi p^{0} \equiv \eta^{0}, \tag{6}$$

$$\mathcal{L}\xi^{i} - \eta^{i} = 0, \tag{7}$$

$$\mathcal{L}\eta^{i} + \chi(\Gamma^{i}_{\mu\nu}p^{\mu}p^{\nu}) = 0. \tag{8}$$

The transformations given by Eqs. (4) to (8) contain not only the geometrical symmetries of the spacetime, which can be obtained from Killing's equation, but also some new symmetry transformations which may be called the non-geometrical symmetries. These new symmetries lead to the non-geometrical constants of motion of the spacetime. So much for generalities. Further progress requires specific assumptions with regard to the metric $g_{\mu\nu}$. Here we study the case of the maximally symmetric spacetime in detail and provide the full symmetries of it.
This case has the maximum number of Killing's vector which is ten (see i.e. Weinberg, 1972). These ten Killing vectors with their corresponding constants of the motion are due to the geometrical symmetries of the spscetime. Here we look for the dynamical symmetries and their corresponding constants of the motion. The metric of the most symmetric spacetime which is known as de Sitter spacetime is:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{\Lambda^{2}}{1 - \Lambda^{2}\eta_{\rho\sigma}x^{\rho}x^{\sigma}}\eta_{\mu\lambda}\eta_{\nu\tau}x^{\lambda}x^{\tau}, \tag{9}$$

where $\eta_{\mu\nu}$ =diag $(-1, 1, 1, 1)$ and Λ^{2} specifies the curvature of the spacetime which may be greater or less than zero. Liouville's operator and Eqs. (6) to (8) for this metric reduce to

$$\mathcal{L} = p^{\mu}\frac{\partial}{\partial x^{\mu}} + \Lambda^{2}m^{2}x^{i}\frac{\partial}{\partial p^{i}}\,. \tag{10}$$

$$\mathcal{L}\xi^o = \chi p^0 \equiv \eta^0, \tag{11}$$
$$\mathcal{L}\xi^i = \eta^i, \tag{12}$$
$$\mathcal{L}\eta^i = \Lambda^2 m^2 \xi^i. \tag{13}$$

Equations (10-13) have two kinds of solutions:
1. There exist ten $\xi^\mu_{(a)}$ $(a = 0, 1, \ldots, 9)$ which satisfy Eqs. (11-13) and are functions of spacetime coordinates only. These ten solutions are:

$$\begin{aligned} &\xi^\mu_{(a)} = A\delta^\mu_a, && (a = 0,1,2,3), \\ &\xi^0_{(3+i)} = 0, && \xi^j_{(3+i)} = -\varepsilon_{ijk}x^k, \\ &\xi^0_{(6+i)} = x^i, && \xi^j_{(6+i)} = x^0\delta^j_i, \end{aligned} \tag{14}$$

where $A = \frac{1}{\Lambda}(1 - \Lambda^2\eta_{\rho\sigma}x^\rho x^\sigma)^{1/2}$ and $B = \eta_{\rho\sigma}x^\rho p^\sigma$. η^i's can be found by Eq. (13).
2. In addition to the above ten Killing's vectors, there are ten new ξ^μ's which satisfy Eqs. (8) but are functions of phase space coordinates. These are as follows:

$$\begin{aligned} &\xi^0_{(10)} = \Lambda^2 Y_{(0)}x^0 - \frac{B}{A}, \qquad \xi^i_{(10)} = \Lambda^2 Y_{(0)}x^i, \\ &\xi^0_{(10+i)} = \Lambda^2 Y_{(i)}x^0, \quad \xi^j_{(10+i)} = \Lambda^2 Y_{(i)}x^j - \frac{B}{A}\delta^j_i, \\ &\xi^0_{(13+i)} = \Lambda^2 Y_{(3+i)}x^0, \ \xi^j_{(13+i)} = \Lambda^2 Y_{(3+i)}x^j - \varepsilon_{ijk}p^k, \\ &\xi^0_{(16+i)} = \Lambda^2 Y_{(6+i)}x^0 + p^i, \\ &\xi^j_{(16+i)} = \Lambda^2 Y_{(6+i)}x^j + p^0\delta^j_i, \end{aligned} \tag{15}$$

where $Y_{(a)}$'s are the ten Killing's constants of the motion given below. η^i's can be found by Eq. (13).

It is known that each symmetry transformation leads to a constant of the motion. Using Eqs. (11-13), one can show that the constants of motion belonging to the geometrical and non-geometrical symmetry transformations are

$$Y_{(a)} = \xi^\lambda_{(a)}p^\lambda - \eta^\lambda_{(a)}x^\lambda \ \ (no\ sum\ on\ \lambda). \tag{16}$$

The first ten constants belonging to the geometrical symmetries are just Killing's constants. These are as follow:

$$Y_{(\mu)} = Ap^\mu + \frac{B}{A}x^\mu, \tag{17}$$
$$Y_{(3+i)} = \varepsilon_{ijk}x^j p^k, \tag{18}$$
$$Y_{(6+i)} = x^i p^0 - x^0 p^i. \tag{19}$$

Using Eq. (12) and the ξ^μ's, and η^μ's given by Eqs. (15), one may obtain the following ten "non-geometrical" contants belonging to the non-geometrical symmetries of de Sitter spacetime:

$$Y_{(10+\mu)} = \frac{B}{A}p^\mu + m^2 A x^\mu, \tag{20}$$
$$Y_{(13+i)} = p^j p^k - \Lambda^2 m^2 x^j x^k, \tag{21}$$
$$Y_{(16+i)} = p^0 p^i - \Lambda^2 m^2 x^0 x^i. \tag{22}$$

Also one may note that the expressions ${p^0}^2 - \Lambda^2 m^2 {x^0}^2$ and ${p^i}^2 - \Lambda^2 m^2 {x^i}^2$ are constants of the motion, but these are proportional to $y_{10}^2 - \Lambda^2 m^2 y_0^2$ and $y_{10+i}^2 - \Lambda^2 m^2 y_i^2$ respectively. Of course not all of the ten Killing constants or the twenty geometrical and dynamical constants are linearly independent.

REFERENCES

Dehghani M. H., Sobouti Y., 1993, Astron. & Astrophys., **275**, 91

Dehghani M. H., Sobouti Y., 1995, Astron. & Astrophys., **299**, 293

Ehler J., 1971, Proceedings of the international summer school of Physics. "Enrico Fermi", Course 47, Edited by R.K. Sachs

Goldstein H., 1980, Clasical Mechanics, Addison Wesley

Maartens R., Maharaj S. D., 1987, Gen. Rel. Grav., **19**, 1217

Sobouti Y., Dehghani M. H., 1992, Astron.& Astrophys., **259**, 128

Weinberg S. 1972, Gravitation and Cosmology, Wiley & Sons, New York