

## FRACTIONAL IDEALS WITH RESPECT TO FINITE ALGEBRAIC EXTENSIONS

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Throughout this paper all rings will be a commutative integral domain with identity. Let  $D$  be an integral domain with quotient field  $K$  and let  $\bar{D}$  be the integral closure in the quotient field of  $D$ . The group of divisibility of  $D$  is the group  $G(D) = K^*/U(D)$  where  $K^*$  is the multiplicative group of  $K$  and  $U(D)$  is the group of units of  $D$ .  $G(D)$  is partially ordered by  $aU(D) \leq bU(D)$  if and only if  $a \mid b$  in  $D$ . Several classes of integral domains  $D$  are completely characterized by the order properties of  $G(D)$ ; for example,  $D$  is a valuation domain if and only if  $G(D)$  is totally ordered. A fractional ideal  $I$  of  $D$  is a  $D$ -submodule of  $K$  for which there is a non-zero element  $x$  of  $D$  with  $xI \subset D$ . The set  $F(D)$  of non-zero fractional ideals of  $D$  forms a commutative monoid under multiplication. A valuation domain  $V$  is said to be discrete if every  $P$ -primary ideal of  $V$  is a power of  $P$ . Note that we are not requiring a discrete valuation domain to have rank one. An  $n$ -dimensional valuation domain  $V$  is discrete if and only if  $G(V) \approx \mathbf{Z}^n$  (as abelian groups) and in this case  $G(V)$  is actually order-isomorphic to the lexicographic direct sum of  $n$  copies of  $\mathbf{Z}$ . For notation and terminology not defined here, the reader is referred to [3] and [5].

In this paper, we show that if  $F(D)$  is finitely generated, and  $L$  is a finite algebraic extension field of  $K$ , then  $F(D')$  is finitely generated where  $D'$  is the integral closure of  $D$  in  $L$ .

We collect in Theorem 0 results from [1] and [2] which will be used throughout this paper without further reference.

**THEOREM 0.** *Let  $\bar{D}$  be the integral closure of  $D$  in the quotient field  $K$  of  $D$ .*

- (1) *If  $F(D)$  is finitely generated, then  $G(D)$  is finitely generated.*

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- (2)  $G(D)$  is finitely generated if and only if  $G(\bar{D})$  is finitely generated and  $\bar{D}/[D : \bar{D}]$  is finite, where  $[D : \bar{D}] = \{d \in D \mid d\bar{D} \subseteq D\}$  is the conductor.
- (3)  $F(D)$  is finitely generated if and only if  $\bar{D}$  is a semiquasilocal Bezout domain such that  $\bar{D}_M$  is a finite rank discrete valuation domain for each maximal ideal  $M$  of  $\bar{D}$  and  $\bar{D}/[D : \bar{D}]$  is finite.
- (4)  $F(D)$  is finitely generated if and only if  $F(\bar{D})$  is finitely generated and  $\bar{D}/[D : \bar{D}]$  is finite.

*Proof.* (1) [2, Theorem 2.3]. (2) [1, Theorem 3]. (3),(4) [2, Theorem 5.3].

**THEOREM 1.** *Let  $D$  be an integral domain with  $F(D)$  finitely generated. Let  $D'$  be the integral closure of  $D$  in a finite algebraic extension field  $L$  of the quotient field  $K$  of  $D$ . Then  $F(D')$  is finitely generated.*

*Proof.* Suppose  $F(D)$  is finitely generated. Then  $F(\bar{D})$  is finitely generated. Since  $D' = \bar{D}'$ , we may assume  $D$  is integrally closed. Since  $G(D)$  is finitely generated,  $G(D')$  is also finitely generated by [4, Theorem 3.13]. Let  $M$  be a maximal ideal of  $D'$  and let  $M \cap D = P$ . Then  $D_P$  is a finite rank discrete valuation domain,  $\text{rank } G(D_P) = \text{rank } G(D'_M)$ , and  $G(D'_M)$  is free. Since  $\dim D = \dim D'$ , we have  $\dim D' = \text{rank } G(D'_M)$ . Hence  $D'_M$  is a finite rank discrete valuation domain and so  $F(D')$  is finitely generated.

If  $D$  is Noetherian, then  $\bar{D}$  is a Krull domain and so  $\bar{D} = \bigcap D_P$ , the intersection being taken over all height-one prime ideals of  $\bar{D}$ . Then  $G(\bar{D})$  can be canonically embedded in the direct sum of the groups  $G(D_P)$ . Thus  $G(\bar{D})$  is free, so  $0 \rightarrow U(\bar{D})/U(D) \rightarrow G(D) \rightarrow G(\bar{D}) \rightarrow 0$  splits. Hence  $G(D) \approx G(\bar{D}) \oplus U(\bar{D})/U(D) \approx \mathbf{Z}^\alpha \oplus U(\bar{D})/U(D)$ , where  $\alpha = \text{rank } G(\bar{D}) \leq |X^{(1)}(\bar{D})|$  and  $X^{(1)}(\bar{D})$  is the set of height-one prime ideals of  $\bar{D}$ . If  $F(D)$  is finitely generated, then  $G(D)$  and  $G(\bar{D})$  are finitely generated and so  $|X^{(1)}(\bar{D})| < \infty$ . Thus  $\bar{D}$  is a semilocal PID. Hence  $D$  is a one-dimensional semilocal domain,  $\text{rank } G(D) = \text{rank } G(\bar{D}) = |X^{(1)}(\bar{D})| \geq |X^{(1)}(D)|$ .

**THEOREM 2.** *Let  $D$  be a one-dimensional semilocal domain such that the residue field of  $D$  with respect to each maximal ideal is finite. If  $F(D')$  is finitely generated and  $D'$  is a finite  $D$ -module, then  $F(D)$  is finitely generated.*

*Proof.* Note that if  $D$  is Noetherian, then  $F(D)$  is finitely generated if and only if  $\bar{D}$  is a semilocal PID and  $\bar{D}/[D : \bar{D}]$  is finite if and only if  $G(D)$  is finitely generated. By [4, Theorem 3.13],  $G(D)$  is finitely generated and hence  $F(D)$  is finitely generated.

### References

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