

## DERIVATIONS OF PRIME RINGS AND BANACH ALGEBRAS

BYUNG-DO KIM

### 1. Introduction.

Throughout,  $R$  represents an associative ring with center  $Z(R)$ . We shall write  $\text{char}(R)$  for the characteristic of ring  $R$ . We write  $[x, y]$  for  $xy - yx$ , and  $\langle x, y \rangle$  for  $xy + yx$ . Then we have  $[\langle y, x \rangle, x] = [\langle x, y \rangle, y]$  for all  $x, y \in R$ . Let  $\text{rad}(A)$  denote the (Jacobson) radical of an algebra  $A$ . Recall that  $R$  is prime if  $aRb = (0)$  implies that either  $a = 0$  or  $b = 0$ . An additive mapping  $D$  from  $R$  to  $R$  is called a derivation if  $D(xy) = D(x)y + xD(y)$  holds for all  $x, y \in R$ . A mapping  $F$  from  $R$  to  $R$  is said to be commuting on  $R$  if  $[F(x), x] = 0$  holds for all  $x \in R$ . In Theorem 1 in Brešar and Vukman [2], they prove the following: Let  $D$  and  $G$  be continuous derivations of a Banach algebra  $A$  such that  $[D^2(x) + G(x), x] \in \text{rad}(A)$  for all  $x \in A$ . Then both  $D$  and  $G$  map  $A$  into  $\text{rad}(A)$ . In Vukman [8] he has proved that in case there exists a nonzero derivation  $D : R \rightarrow R$ , where  $R$  is a prime ring of characteristic different from 2, 3, such that the mapping  $x \mapsto [D(x), x]$  is commuting on  $R$ ,  $R$  is commutative. We are to prove that if there is a derivation  $D$  on a noncommutative prime ring  $R$  such that  $[\langle Dx, x \rangle, x] = 0$  for all  $x \in R$  and  $\text{char}(R) \neq 2$ , then we have  $D = 0$ . And making use of the result, we can prove that the range of a continuous linear Jordan derivation on a noncommutative Banach algebra  $A$  is contained in the radical under the condition  $[\langle Dx, x \rangle, x] \in \text{rad}(A)$  for all  $x \in A$ . For further references we refer to [1, 3, 4, 9].

### 2. Main Results.

---

Received November 2, 1994.

**THEOREM 2.1.** *Let  $R$  be a noncommutative prime ring with  $\text{char}(R) \neq 2$ . Suppose there exists a derivation  $D : R \rightarrow R$  such that  $x \mapsto \langle Dx, x \rangle$  is commuting on  $R$ . Then we have  $D = 0$ .*

*Proof.* We define a mapping  $B(\cdot, \cdot) : R \times R \rightarrow R$  by

$$B(x, y) = [Dx, y] + [Dy, x], \quad x, y \in R.$$

$B(y, x) = B(x, y)$  for all  $x, y \in R$  and additive in both arguments. We have the relation

$$B(xy, z) = B(x, z)y + xB(y, z) + (Dx)[y, z] + [x, z]Dy,$$

for all  $x, y, z \in R$ . We also define the following:  $e(x) = 2Dx$ ,  $f(x) = [e(x), x]$ ,  $x \in R$ . Then  $f(x) = B(x, x)$ ,  $x \in R$ . And it follows that

$$f(x + y) = f(x) + f(y) + 2B(x, y), \quad x, y \in R.$$

The assumption of the theorem can now be written in the form

$$[\langle Dx, x \rangle, x] = 0, \quad x, y \in R. \quad (1)$$

From (1), one obtains

$$f(x)x + xf(x) = 0, \quad x \in R. \quad (2)$$

The linearization of (2) gives

$$\begin{aligned} f(x)y + yf(x) + 2B(x, y)x + 2xB(x, y) + (f(y)x + xf(y) \\ + 2B(x, y)y + 2yB(x, y)) = 0, \quad x, y \in R. \end{aligned} \quad (3)$$

Replacing  $-y$  for  $y$  in (3), we have

$$\begin{aligned} -f(x)y - yf(x) - 2B(x, y)x - 2xB(x, y) + (f(y)x + xf(y) \\ + 2B(x, y)y + 2yB(x, y)) = 0, \quad x, y \in R. \end{aligned} \quad (4)$$

Subtracting (4) from (3), using the condition  $\text{char}(R) \neq 2$  we obtain

$$f(x)y + yf(x) + 2B(x, y)x + 2xB(x, y) = 0, \quad x, y \in R. \quad (5)$$

Substituting  $yx$  for  $y$  in (5), it follows that

$$\begin{aligned} f(x)yx + yf(x)x + 2B(x, y)x^2 + 2xB(x, y)x + yf(x)x + yxf(x) \\ + 2xyf(x) + 2xyf(x) + [y, x]e(x)x + x[y, x]e(x) = 0, \quad x, y \in R. \end{aligned} \quad (6)$$

Combining (2), (5) with (6), we arrive at

$$2xyf(x) + [y, x]e(x)x + x[y, x]e(x) = 0, \quad x, y \in R. \quad (7)$$

Replacing  $e(x)y$  for  $y$  in (7), one obtains

$$\begin{aligned} 2xe(x)yf(x) + e(x)[y, x]e(x)x + f(x)ye(x)x + xe(x)[y, x]e(x) \\ + xf(x)ye(x) = 0, \quad x, y \in R. \end{aligned} \quad (8)$$

Left multiplication of (7) by  $e(x)$  leads to

$$2e(x)xyf(x) + e(x)[y, x]e(x)x + e(x)x[y, x]e(x) = 0, \quad x, y \in R. \quad (9)$$

Subtracting (8) from (9), we arrive at

$$\begin{aligned} 2f(x)yf(x) - f(x)ye(x)x + f(x)[y, x]e(x) \\ - xf(x)ye(x) = 0, \quad x, y \in R. \end{aligned} \quad (10)$$

Since  $f(x)[y, x]e(x) = f(x)yxe(x) - f(x)xye(x)$ ,  $x, y \in R$ , (10) reduces to

$$f(x)yf(x) - (f(x)x + xf(x))ye(x) = 0, \quad x, y \in R. \quad (11)$$

Hence, combining (2) with (11) we arrive at

$$f(x)yf(x) = 0, \quad x, y \in R. \quad (12)$$

By primeness of  $R$  it follows from (12) that

$$f(x) = 0, \quad x \in R.$$

Therefore, by Lemma 3 in [6] we have

$$Dx = 0, \quad x \in R.$$

The proof of the theorem is complete.  $\square$

**THEOREM 2.2.** *Let  $A$  be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation  $D : A \rightarrow A$  such that  $[\langle Dx, x \rangle, x] \in \text{rad}(A)$ ,  $x \in A$ . Then we have  $DA \subseteq \text{rad}(A)$ .*

*Proof.* Let  $D : A \rightarrow A$  be a continuous linear Jordan derivation. Then by Lemma 3.2 in [7],  $DP \subseteq P$ . Hence we have  $D(\text{rad}(A)) \subseteq \text{rad}(A)$ . And so, we may assume that  $A$  is semisimple. Now since  $DP \subseteq P$ , we can introduce a Jordan derivation  $D_P : A/P \rightarrow A/P$  by  $D_P(x + P) = Dx + P$ ,  $x \in A$ . When  $P$  is a primitive ideal,  $A/P$  is a prime algebra. Thus by Herstein's result  $D_P$  is a derivation. And the assumption of the theorem gives  $[\langle D_P(x + P), x + P \rangle, x + P] = 0$ ,  $x \in A$ . Hence in case  $A/P$  is noncommutative, we have  $D_P = 0$  by Theorem 2.1. It is sufficient to prove that  $D_P = 0$  when  $A/P$  is commutative. Johnson [5] have proved that any linear derivation on a semisimple Banach algebra is continuous. By this result and Singer-Wermer's Theorem, we obtain that there is the only zero derivation on commutative Banach algebras. Hence in this case we have  $D_P = 0$  as well. Consequently, this means that  $DA \subseteq \text{rad}(A)$ . The proof of the theorem is complete.  $\square$

**COROLLARY 2.3.** *Let  $A$  be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation  $D : A \rightarrow A$  such that  $[D(x^2), y] \in \text{rad}(A)$ ,  $x, y \in A$ . Then we have  $DA \subseteq \text{rad}(A)$ .*

*Proof.* All the assumptions of Theorem 2.2 are fulfilled.  $\square$

**THEOREM 2.4.** *Let  $A$  be a noncommutative semisimple Banach algebra. Suppose there exists a linear Jordan derivation  $D : A \rightarrow A$  such that  $D(x^2) \in Z(A)$ ,  $x \in A$ . Then we have  $D = 0$ .*

*Proof.* The proof goes through in the same fashion as the proof of Theorem 2.2 except the fact that at the beginning of the proof we have to use the fact that any linear Jordan derivation on a semisimple Banach algebra is continuous (see [1, Theorem 6]).  $\square$

## References

- [1] M. Brešar, *Jordan derivations on semiprime rings*, Proc. Amer. Math. Soc. **104** (1988), 1003–1006.
- [2] M. Brešar and J. Vukman, *Derivations of noncommutative Banach algebras*, Arch. Math. **59** (1992), 363–370.

- [3] J. Cusack, *Jordan derivations on rings*, Proc. Amer. Math. Soc. **53** (1975), 321–324.
- [4] I.N. Herstein, *Jordan derivations on prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1104–1110.
- [5] B.E. Johnson, *Continuity of derivations on commutative Banach algebras*, Amer. J. Math. **91** (1969), 1–10.
- [6] E. Posner, *Derivations in prime rings*, Proc. American Mathematic Soc. **8** (1957), 1093–1100.
- [7] A.M. Sinclair, *Jordan homomomorphisms and derivations on semisimple Banach algebras*, Proc. Amer. Math. Soc. **24** (1970), 209–214.
- [8] J. Vukman, *Commuting and centralizing mappings in prime rings*, Proc. Amer. Math. Soc. **109** No.1 (1990), 47–52.
- [9] J. Vukman, *On derivations in prime rings and Banach algebras*, Proc. Amer. Math. Soc. **116** No. 4 (1992), 877–884.

Department of Mathematics  
Kangnung National University  
Kangnung, 210–702, Korea