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# Analysis of Priority Systems with a Mixed Service Discipline

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**Abstract**

We investigate a multiclass priority system with a mixed service discipline, and propose a new approach to the analysis of performance measures (mean waiting times) of the system.

Customers are preferentially served in the order of priority. The service discipline at each station is either gated or exhaustive discipline.

We formulate mean waiting times as functions on the state of the system. We first consider the system at an arbitrary system state to obtain explicit formulae for the mean waiting times, and then derive their steady state values by using the property of Poisson arrivals to see time averages and the generalized Little's formula.

## 1. Introduction

Priority queueing systems have been investigated enormously; Jaiswal [4] and Kleinrock [5] have treated them extensively. Wolff [10] summarized standard methods for analyzing priority queues. Various scheduling algorithms of priority queues are investigated by Takagi [7]. These researches have been carried out by assuming that the service discipline is the same for all stations in the system. Takagi [8] considered a multiple-stations polling system with a mix of exhaustive and gated service discipline, and derived the mean waiting times. The service discipline of his model is cyclic order. Harrison [1] has formulated some system

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performance measures as functions on the state space of the system. This method is used for an optimization problem in [2]. In this paper, we investigate a multiclass priority system with a mixture of gated and exhaustive service disciplines, and propose a new approach to the analysis of performance measures of the system. We first define an appropriate system state and a stochastic process which represents an evolution of the system, and then formulate system performance measures (cost functions) such as the mean waiting time as functions of the system state. Next, we obtain explicit formulae for the cost functions at an arbitrary state. Finally we derive their steady state values. This comes from PASTA (Poisson arrivals see time averages) and the generalized Little's formula. Since these values are expressed in matrix forms, we can easily construct an algorithm for yielding these values. As preliminaries, we analyze busy periods of the system, the expected value of works (service times) of customers who complete their service at a station during a busy period, and the number of waiting customers at each station at the completion epoch of a service period defined as the time interval that the server continuously serves a station.

## 2. Model and Notation

We consider a multiclass priority queueing system. There are $J$ stations, indexed as $1, 2, \ldots, J$. We assume that each station has an infinite capacity. Customers arrive at station $j$ according to a Poisson process with rate $\lambda_j$ $(j = 1, \ldots, J)$. Let $\lambda = \sum_{j=1}^{J} \lambda_j$. Customers are served by a single server in the order of priority according to a predetermined *scheduling discipline* for which station $i$ has priority over station $j$ if $i < j$. Each customer in station $j$, who is called a *class $j$ customer*, requires an independent random service $S_j$. The service discipline at each station is either gated or exhaustive discipline. If the server selected a station with gated discipline, he continues to serve only customers who are waiting at the station when it is selected in arrival order. If the server selected a station with exhaustive discipline, he will continue to serve the station in arrival order until the station is emptied. We consider a nonpreemptive discipline, that is, a customer once beginning its service is not interrupted until his current service is completed, even if customers with higher priority arrive. We often use superscripts $G$ and $E$ to distinguish gated discipline and exhaustive discipline, respectively. The server utilization $\rho_j$ of customers of class $j$ and the aggregate server utilization $\rho_j^+$ of customers of classes 1 through $j$ are given by

$$\rho_j = \lambda_j E[S_j],$$
$$\rho_j^+ = \sum_{i=1}^{j} \rho_i.$$

Throughout the paper we assume that $\rho = \sum_{i=1}^{J} \rho_i < 1$. The time interval which the server continuously serves station $j$ is called *a service period of station* $j$. Specifically, if we would like to specify that a specific customer is scheduled to serve during a period, we call the period his service period. Let $\Pi = \{1, \ldots, J\}$ be a set of service periods (or stations). $\Pi_G$ denotes a set of stations with gated discipline, and $\Pi_E$ a set of stations with exhaustive discipline. Let $\kappa$ denote the station being served by the server (current service period). It is assumed that $\kappa = 0$ when the system is empty. Let $r$ be a *remaining service time* of the customer being served. Since all customers in a station with gated discipline who arrive during a current period are set aside to be served at next selecting, we specify the number $g$ of customers scheduled to be served during the current service period, who are called *customers within the gate*. $g$ does not count the customer being served. The customers who are not currently served nor scheduled to serve during the current service period are called *waiting customers*. Let $\mathcal{R}, \mathcal{R}_+, \mathcal{I}_+$ be respectively a set of real numbers, a set of nonnegative real numbers, and a set of nonnegative integers. The number of waiting customers at station $j$ is denoted by $n_j$ and their vector is denoted by $\mathbf{n} \equiv (n_1, \ldots, n_J) \in \mathcal{I}_+^J$. Let $\kappa(t) \in \mathcal{R}_+$ denotes a station being served at time $t$, and $r(t) \in \mathcal{R}_+$ a remaining service time of the customer being served at time $t$. The number of customers within the gate at time $t$ is denoted by $g(t) \in \mathcal{I}_+$, and the number of waiting customers in station $i$ at time $t$ is denoted by $n_i(t)$. Let a vector $\mathbf{n}(t) \equiv (n_1(t), \ldots, n_J(t)) \in \mathcal{I}_+^J$. The processes $\{\kappa(t) : t \geq 0\}, \{r(t) : t \geq 0\}$ and $\{g(t) : t \geq 0\}$ are right continuous with left-hand limits, except for customer's arrival epochs at which these processes are left continuous with right-hand limits. The process $\{\mathbf{n}(t) : t \geq 0\}$ is left continuous with right-hand limits. Let us assume that customers are numbered in the order of their arrivals. We consider an $e^{th}$ customer arrives at one of the stations at epoch $\sigma^e$. We will specify informations of the system in order to operate it according to a predetermined scheduling discipline. Let $l_{im}(t) \in \mathcal{R}_+ \cup \{\infty\}$ be an information of a customer in $m^{th}$ position of the queue of station $i$ at time $t$. If the $e^{th}$ customer is in $m^{th}$ position of the queue of station $i$ at time $t$, $l_{im}(t) = \sigma^e$. If there is not a class $i$ customer in $m^{th}$ position of its queue at time $t$, let $l_{im}(t) = \infty$. The *customer list* is a set of these informations such that $L(t) = (l_{im}(t) : i = 1, \ldots, J$ and $m = 1, 2, \ldots)$. Let us consider *transition epochs* of these processes such as customer's arrival epochs and service completion epochs. Let $X(t)$ denote a station where the last transition before $t$ occurs $(t \geq 0)$. $X(t)$ is right continuous with left-hand limits. Then we define a stochastic process $\mathcal{Q} = \{\mathbf{Y}(t) = (X(t), \kappa(t), r(t), g(t), \mathbf{n}(t), L(t)) : t \geq 0\}$ that represents an evolution of the system. For any scheduling discipline defined above, $\mathcal{Q}$ embeds a Markov process with a stationary transition probability whose transition epochs consist of customer's arrival epochs and service completion epochs. Possible values of $\mathbf{Y}(t)$ $(t \geq 0)$ are called *system states* (or sates). The state space of $\mathcal{Q}$ is denoted by $\mathcal{E}$. We focus on the $e^{th}$ arrival customer at epoch $\sigma^e$ where the system state is $\mathbf{Y} = (k, \kappa, r, g, \mathbf{n}, L) \in \mathcal{E}$. Let

$$C^e_{Wj}(t) \equiv \begin{cases} 1, & \text{if the } e^{th} \text{ arriving customer stays at station } j \\ & \text{as a waiting customer at time } t, \\ 0, & \text{otherwise,} \end{cases} \tag{2.1}$$

$$C^e_{Gj}(t) \equiv \begin{cases} 1, & \text{if the } e^{th} \text{ arriving customer stays within the gate} \\ & \text{at station } j \text{ at time } t, \\ 0, & \text{otherwise,} \end{cases} \tag{2.2}$$

for $t \geq 0$, $j = 1, \ldots, J$ and $e = 1, 2, \ldots$.

We would like to derive three types of cost functions defined as follows:

$$W_j(\mathbf{Y}, e) \equiv E\left[\int_{\sigma^e}^{\infty} C^e_{Wj}(t)dt | \mathbf{Y}(\sigma^e) = \mathbf{Y}\right], \tag{2.3}$$

Then $W_j(\mathbf{Y}, e)$ denotes a mean waiting time of an $e^{th}$ class $j$ customer spent at station $j$ as a waiting customer given that the system is in state $\mathbf{Y}$ at his arrival epoch. $W_j(\mathbf{Y}, e) \equiv 0$ for $\mathbf{Y} = (k, \kappa, r, g, \mathbf{n}, L) \in \mathcal{E}$ such that $k \neq j$.

$$H_j(\mathbf{Y}, e, i) \equiv E\left[\int_{\sigma^e}^{\infty} C^e_{Wj}(t) 1\{\kappa(t) = i\} dt | \mathbf{Y}(\sigma^e) = \mathbf{Y}\right], \quad i = 1, \ldots, J \tag{2.4}$$

where, for any event $\mathcal{K}$, $\mathbf{1}\{\mathcal{K}\}$ equals 1 if event $\mathcal{K}$ occurs, and equals 0 otherwise. Then $H_j(\mathbf{Y}, e, i)$ denotes a mean waiting time of an $e^{th}$ class $j$ customer spent as a waiting customer during the system is in the service period of station $i$ given that the system is in state $\mathbf{Y}$ at his arrival epoch. $H_j(\mathbf{Y}, e, i) \equiv 0$ for $\mathbf{Y} = (k, \kappa, r, g, \mathbf{n}, L) \in \mathcal{E}$ such that $k \neq j$.

$$G_j(\mathbf{Y}, e) \equiv E\Big[\int_{\sigma^e}^{\infty} C^e_{G_j}(t)dt|\mathbf{Y}(\sigma^e) = \mathbf{Y}\Big], \tag{2.5}$$

Then $G_j(\mathbf{Y}, e)$ denotes a mean waiting time of an $e^{th}$ class $j$ customer spent within the gate given that the system is in state $\mathbf{Y}$ at his arrival epoch. $G_j(\mathbf{Y}, e) \equiv 0$ for $\mathbf{Y} = (k, \kappa, r, g, \mathbf{n}, L) \in \mathcal{E}$ such that $k \neq j$.

## 3. Busy period and Work

The cost functions defined in the last section will be shown to be closely related to busy periods. So we define quantities related to busy periods. We select any set of customers initially in the system and compose a set $\mathcal{C} = \mathcal{C}(\mathbf{Y})$ of these customers. For example, if a class $i$ customer in the $m^{th}$ position of its queue is in $\mathcal{C}$, then $(i, m) \in \mathcal{C}$. His service time is often denoted by $S_i^m$, which has the same distribution as $S_i$. A set of customers who are initially in the system and are not in the set $\mathcal{C}$ is denoted by $\mathcal{C}^c = \mathcal{C}^c(\mathbf{Y}; \mathcal{C})$.

Let $B^j(v)$ be a busy period starting with an initial service time (exceptional service time) $v$ until the first epoch when the system is cleared of the customer with service $v$ and all customers from classes 1 through $j$. Let $B^j(\mathbf{Y}; \mathcal{C})$ be a busy period starting with state $\mathbf{Y}$ until the first epoch when the system is cleared of the customer being served, the customers in $\mathcal{C}$, and the customers from classes 1 through $j$ except for customers in $\mathcal{C}^c$. We will call $B^j(\mathbf{Y}; \mathcal{C})$ a *class $j$ busy period* initiated with $\{\mathbf{Y}; \mathcal{C}\}$. It can be shown that these busy periods are composed of sub-busy periods initiated with each customer in $\mathcal{C}$. Its expected value is easily obtained by the usual method [10]. Then we have

$$E[B^j(\mathbf{Y}; \mathcal{C})] = \frac{r + \sum\sum_{(i,m)\in\mathcal{C}} E[S_i^m]}{1 - \rho_j^+}. \tag{3.1}$$

Now we define $V_l^j(S_i)$ $(i, l = 1, \ldots, j)$ be the total amount of works (service times) of customers who complete their services at station $l$ during a busy period $B^j(S_i)$. Then it can be easily shown that

$$E[V_l^j(S_i)] = \begin{cases} \sum_{k=1}^{j} \lambda_k E[S_i] E[V_l^j(S_k)], & i = 1,\dots,j \text{ and } i \neq l, \\ E[S_l] + \sum_{k=1}^{j} \lambda_k E[S_l] E[V_l^j(S_k)], & i = l, \end{cases} \tag{3.2}$$

for $l = 1,\dots,j$ [3]. Further we define $V_l^j(v)$ $(l = 1,\dots,J)$ be the total amount (except for exceptional service $v$) of works of customers who complete their services at station $l$ during a busy period $B^j(v)$. Then we have

$$E[V_l^j(v)] = \sum_{k=1}^{j} \lambda_k v E[V_l^j(S_k)], \quad l = 1,\dots,j. \tag{3.3}$$

By defining constants:

$$\xi_l^j \equiv \begin{cases} \sum_{k=1}^{j} \lambda_k E[V_l^j(S_k)] = \rho_l/(1-\rho_j^+), & l = 1,\dots,j, \\ 0, & l = j+1,\dots,J, \end{cases} \tag{3.4}$$

we have

$$E[V_l^j(v)] = v\xi_l^j, \quad l = 1,\dots,J. \tag{3.5}$$

Finally let $V_l^j(\mathbf{Y};\mathcal{C})$ $(l = 1,\dots,j)$ be the total amount of works of customers who complete their services at station $l$ during a busy period $B^j(\mathbf{Y};\mathcal{C})$. Then we have

$$E[V_l^j(\mathbf{Y};\mathcal{C})] = \begin{cases} E[V_l^j(r)] + r + \sum\sum_{(i,m)\in\mathcal{C}} E[V_l^j(S_i^m)] + \sum_{(l,m)\in\mathcal{C}} E[S_l^m], & l = \kappa, \\ E[V_l^j(r)] + \sum\sum_{(i,m)\in\mathcal{C}} E[V_l^j(S_i^m)] + \sum_{(l,m)\in\mathcal{C}} E[S_l^m], & l \neq \kappa, \end{cases} \tag{3.6}$$

$$= \begin{cases} r + \xi_l^j \left\{ r + \sum\sum_{(i,m)\in\mathcal{C}} E[S_i^m] \right\} + \sum_{(l,m)\in\mathcal{C}} E[S_l^m], & l = \kappa, \\ \xi_l^j \left\{ r + \sum\sum_{(i,m)\in\mathcal{C}} E[S_i^m] \right\} + \sum_{(l,m)\in\mathcal{C}} E[S_l^m], & l \neq \kappa, \end{cases} \tag{3.7}$$

for $l = 1,\dots,J$. It can be easily shown that

$$E[B^j(\mathbf{Y};\mathcal{C})] = \sum_{l=1}^{J} E[V_l^j(\mathbf{Y};\mathcal{C})]. \tag{3.8}$$

These values are used to obtain explicit formulae of the cost functions.

## 4. System at arbitrary system states

In this section, we derive formulae of the cost functions $W_j(\mathbf{Y}, e)$, $H_j(\mathbf{Y}, e, i)$ and $G_j(\mathbf{Y}, e)$. Since these cost functions are equal to 0 for $k \neq j$, we consider them for the case $k = j$. Let $R(\mathbf{Y})$ be a time to complete the current service period which has being executed at the arrival epoch $\sigma^e$. Let $s = \sigma^e + R(\mathbf{Y})$ be the completion epoch of the current service period. Then $n_i(s)$ denotes the number of waiting customers at station $i$ ($i = 1, \dots, J$) at the completion epoch of the current service period. Then we have

$$E[R(\mathbf{Y})] = \begin{cases} r + gE[S_\kappa], & \kappa \in \Pi_G, \\ \{r + gE[S_\kappa]\}/(1 - \rho_\kappa), & \kappa \in \Pi_E, \end{cases} \tag{4.1}$$

and we have

$$E[n_i(s)] = \begin{cases} n_i + 1_{ij} + \lambda_i E[R(\mathbf{Y})], & \kappa \in \Pi_G, \\ \{n_i + 1_{ij} + \lambda_i E[R(\mathbf{Y})]\}(1 - 1_{\kappa i}), & \kappa \in \Pi_E, \end{cases} \tag{4.2}$$

for $i = 1, \dots, J$, where $\mathbf{1}_{ki}$ equals 1 if $k = i$, and equals 0 if $k \neq i$. Note that these expected values are linear functions of components $r, g$ and $\mathbf{n}$ of state $\mathbf{Y}$. Let $\mathcal{C}_{j-1} = \mathcal{C}_{j-1}(\mathbf{Y})$ denote a set of customers composed of all waiting customers from classes 1 through $j-1$ at the epoch $s$ ($\mathcal{C}_0 = \emptyset$).

### Gated disciplines.

We derive expressions of the cost functions for station $j$ with gated discipline ($j \in \Pi_G$). The waiting time $W_j^G(\mathbf{Y}, e)$ is composed of the mean values of $R(\mathbf{Y})$ and a class $j-1$ busy period initiated with $\{\mathbf{Y}(s); \mathcal{C}_{j-1}\}$, regardless of the service disciplines adopted by the other stations. Hence we have

$$\begin{aligned} W_j^G(\mathbf{Y}, e) &= E[R(\mathbf{Y}) + B^{j-1}(\mathbf{Y}(s); \mathcal{C}_{j-1})] \\ &= E[R(\mathbf{Y})] + \frac{\sum_{i=1}^{j-1} E[n_i(t)]E[S_i]}{1 - \rho_{j-1}^+} \\ &= \begin{cases} \left\{E[R(\mathbf{Y})] + \sum_{l=1}^{j-1} n_l E[S_l]\right\}/(1 - \rho_{j-1}^+), & \kappa \in \Pi_G, \\ \left\{E[R(\mathbf{Y})](1 - \rho_\kappa \sum_{l=1}^{j-1} 1_{\kappa l}) + \sum_{l=1}^{j-1} n_l E[S_l] \right. & \\ \quad \left. - n_\kappa E[S_\kappa] \sum_{l=1}^{j-1} 1_{\kappa l}\right\}/(1 - \rho_{j-1}^+), & \kappa \in \Pi_E. \end{cases} \end{aligned} \tag{4.3}$$

His waiting time $H_j^G(\mathbf{Y},e,i)$ is equal to the total amount of mean service times completed at station $i$ during the class $j-1$ busy period initiated with $\{\mathbf{Y}(s);\mathcal{C}_{j-1}\}$ if $\kappa \neq i$, and the amount plus mean length of the remaining service period $\kappa$ if $\kappa = i$. Hence we have

$$H_j^G(\mathbf{Y},e,i) \;=\; E[R(\mathbf{Y})]\mathbf{1}_{\kappa i} + E[V_i^{j-1}(\mathbf{Y}(s);\mathcal{C}_{j-1})], \quad i = 1,\ldots,J. \tag{4.4}$$

From (3.7), we have

$$E[V_i^{j-1}(\mathbf{Y}(s);\mathcal{C}_{j-1})] = \begin{cases} \xi_i^{j-1}\left\{\sum_{l=1}^{j-1} E[n_l(s)]E[S_l]\right\} + E[n_i(s)]E[S_i], & i < j, \\ 0, & i \geq j. \end{cases} \tag{4.5}$$

Hence we have

$$H_j^G(\mathbf{Y},e,i) \;=\; \begin{cases} E[R(\mathbf{Y})]\left\{\mathbf{1}_{\kappa i} + \xi_i^{j-1}\right\} \\ \quad + \sum_{l=1}^{j-1} n_l E[S_l]\xi_i^{j-1} + n_i E[S_i], & i < j \text{ and } \kappa \in \Pi_G, \\ E[R(\mathbf{Y})]\left\{\mathbf{1}_{\kappa i} + \xi_i^{j-1} - \rho_\kappa\left(\xi_i^{j-1}\sum_{l=1}^{j-1}\mathbf{1}_{\kappa l} + \mathbf{1}_{\kappa i}\right)\right\} \\ \quad + \sum_{l=1}^{j-1} n_l E[S_l]\xi_i^{j-1} + n_i E[S_i] \\ \quad - n_\kappa E[S_\kappa]\left(\xi_i^{j-1}\sum_{l=1}^{j-1}\mathbf{1}_{\kappa l} + \mathbf{1}_{\kappa i}\right), & i < j \text{ and } \kappa \in \Pi_E, \\ E[R(\mathbf{Y})]\mathbf{1}_{\kappa i}, & i \geq j. \end{cases} \tag{4.6}$$

On the other hand, once station $j$ is selected, all customers in station $j$ in front of the $e^{th}$ customer are continuously served in the arrival order. Hence we have

$$G_j^G(\mathbf{Y},e) = n_j E[S_j]. \tag{4.7}$$

**Exhaustive disciplines.**

We derive expressions of the cost functions for station $j$ with exhaustive discipline ($j \in \Pi_E$). If $\kappa \neq j$, the waiting time $W_j^E(\mathbf{Y},e)$ is composed of the mean values of $R(\mathbf{Y})$ and a class $j-1$ busy period initiated with $\{\mathbf{Y}(s);\mathcal{C}_{j-1}\}$, regardless of the service disciplines adopted by the other stations. If $\kappa = j$, his waiting time is equal to 0, because the customer immediately enters within the gate. Then

$$W_j^E(\mathbf{Y},e) = \begin{cases} 0, & \kappa = j, \\ E[R(\mathbf{Y}) + B^{j-1}(\mathbf{Y}(s);\mathcal{C}_{j-1})], & \kappa \neq j. \end{cases} \tag{4.8}$$

Hence we have

$$W_j^E(\mathbf{Y},e)=\begin{cases}0, & \kappa=j,\\ \left\{E[R(\mathbf{Y})]+\sum_{l=1}^{j-1} n_l E[S_l]\right\}/(1-\rho_{j-1}^+), & \kappa\neq j \text{ and } \kappa\in\Pi_G,\\ \left\{E[R(\mathbf{Y})](1-\rho_\kappa\sum_{l=1}^{j-1}\mathbf{1}_{\kappa l})+\sum_{l=1}^{j-1} n_l E[S_l] \right. & \\ \quad \left. -n_\kappa E[S_\kappa]\sum_{l=1}^{j-1}\mathbf{1}_{\kappa l}\right\}/(1-\rho_{j-1}^+), & \kappa\neq j \text{ and } \kappa\in\Pi_E.\end{cases} \tag{4.9}$$

If $\kappa\neq j$ and $i\neq j$, his waiting time $H_j^E(\mathbf{Y},e,i)$ is equal to the total amount of mean service times completed at station $i$ before his service period begins, otherwise it equals to 0. Thus we have

$$H_j^E(\mathbf{Y},e,i) = \begin{cases}0, & \kappa=j \text{ and } i=1,\ldots,J,\\ 0, & \kappa\neq j \text{ and } i=j,\\ E[R(\mathbf{Y})]\mathbf{1}_{\kappa i}+E[V_i^{j-1}(\mathbf{Y}(s);\mathcal{C}_{j-1})], & \kappa\neq j \text{ and } i\neq j.\end{cases} \tag{4.10}$$

From (3.7), we have

$$\begin{aligned}&E[V_i^{j-1}(\mathbf{Y}(s);\mathcal{C}_{j-1})]\\ &= \begin{cases}\xi_i^{j-1}\left\{\sum_{l=1}^{j-1}E[n_l(s)]E[S_l]\right\}+E[n_i(s)]E[S_i], & i<j,\\ 0, & i\geq j.\end{cases}\end{aligned} \tag{4.11}$$

Hence we have

$$H_j^E(\mathbf{Y},e,i) = 0, \quad i=1,\ldots,J, \tag{4.12}$$

for $\kappa=j$, and we have

$$H_j^E(\mathbf{Y},e,i) = \begin{cases}E[R(\mathbf{Y})]\left\{\mathbf{1}_{\kappa i}+\xi_i^{j-1}\right\} & \\ \quad +\sum_{l=1}^{j-1} n_l E[S_l]\xi_i^{j-1}+n_iE[S_i], & i<j \text{ and } \kappa\in\Pi_G,\\ E[R(\mathbf{Y})]\left\{\mathbf{1}_{\kappa i}+\xi_i^{j-1}-\rho_\kappa\left(\xi_i^{j-1}\sum_{l=1}^{j-1}\mathbf{1}_{\kappa l}+\mathbf{1}_{\kappa i}\right)\right\} & \\ \quad +\sum_{l=1}^{j-1} n_l E[S_l]\xi_i^{j-1}+n_iE[S_i] & \\ \quad -n_\kappa E[S_\kappa]\left(\xi_i^{j-1}\sum_{l=1}^{j-1}\mathbf{1}_{\kappa l}+\mathbf{1}_{\kappa i}\right), & i<j \text{ and } \kappa\in\Pi_E,\\ 0, & i=j,\\ E[R(\mathbf{Y})]\mathbf{1}_{\kappa i}, & i>j,\end{cases} \tag{4.13}$$

for $\kappa \neq j$. Similarly in the case of the gated discipline, once station $j$ is selected, all customers at station $j$ in front of the $e^{th}$ customer are continuously served in the arrival order. Hence we have

$$G_j^E(\mathbf{Y},e) = \begin{cases} r + gE[S_j], & \kappa = j, \\ n_j E[S_j], & \kappa \neq j. \end{cases} \tag{4.14}$$

We generalize these results to obtain steady state values of the cost functions. The cost functions $W_j(\cdot), H_j(\cdot,i)$ and $G_j(\cdot)$ are shown to be linear functions of $r, g$ and $\mathbf{n}$ of state $\mathbf{Y}$ for any given service period $\kappa$ because the functions $E[R(\mathbf{Y})]$ and $E[n_i(s)]$ are also linear functions of $r, g$ and $\mathbf{n}$ of state $\mathbf{Y}$ for any given service period $\kappa$. Then by appropriately choosing coefficients, we can express as follows. For $j \in \Pi_G$,

$$W_j(\mathbf{Y},e) = \begin{cases} r\varphi_j(\kappa) + g\psi_j(\kappa) + \sum_{l=1}^{j} n_l\phi_{lj}, & \kappa \in \Pi_G, \\ r\varphi_j(\kappa) + g\psi_j(\kappa) + \sum_{l=1}^{j} n_l\phi_{lj} + n_\kappa \phi_{Wj}(\kappa), & \kappa \in \Pi_E, \end{cases} \tag{4.15}$$

$$H_j(\mathbf{Y},e,i) = \begin{cases} r\varphi_j(\kappa,i) + g\psi_j(\kappa,i) + \sum_{l=1}^{j} n_l\phi_{lj}(i), & \kappa \in \Pi_G, \\ r\varphi_j(\kappa,i) + g\psi_j(\kappa,i) + \sum_{l=1}^{j} n_l\phi_{lj}(i) + n_\kappa \phi_{Hj}(\kappa,i), & \kappa \in \Pi_E, \end{cases} \tag{4.16}$$

$$G_j(\mathbf{Y},e) = n_j\zeta_j. \tag{4.17}$$

For $j \in \Pi_E$,

$$W_j(\mathbf{Y},e) = \begin{cases} r\varphi_j(\kappa) + g\psi_j(\kappa) + \sum_{l=1}^{j} n_l\phi_{lj}, & \kappa \neq j, \kappa \in \Pi_G, \\ r\varphi_j(\kappa) + g\psi_j(\kappa) + \sum_{l=1}^{j} n_l\phi_{lj} & \\ \qquad + n_\kappa \phi_{Wj}(\kappa), & \kappa \neq j, \kappa \in \Pi_E, \\ 0, & \kappa = j, \end{cases} \tag{4.18}$$

$$H_j(\mathbf{Y},e,i) = \begin{cases} r\varphi_j(\kappa,i) + g\psi_j(\kappa,i) + \sum_{l=1}^{j} n_l\phi_{lj}(i), & \kappa \neq j, \kappa \in \Pi_G, \\ r\varphi_j(\kappa,i) + g\psi_j(\kappa,i) + \sum_{l=1}^{j} n_l\phi_{lj}(i) & \\ \qquad + n_\kappa \phi_{Hj}(\kappa,i), & \kappa \neq j, \kappa \in \Pi_E, \\ 0, & \kappa = j, \end{cases} \tag{4.19}$$

$$G_j(\mathbf{Y},e) = \begin{cases} n_j\zeta_j, & \kappa \neq j, \\ r + g\theta_j, & \kappa = j. \end{cases} \tag{4.20}$$

Of course, each scheduling algorithm has its own coefficients. For simplicity, we define the following vectors:

$$\mathbf{w}_j \equiv (\phi_{1j}, \ldots, \phi_{jj}, 0, \ldots, 0)' \in \mathcal{R}^{J\times 1}, \tag{4.21}$$

$$\mathbf{h}_j(i) \equiv (\phi_{1j}(i), \ldots, \phi_{jj}(i), 0, \ldots, 0)' \in \mathcal{R}^{J\times 1}, \tag{4.22}$$

$$\mathbf{g}_j \equiv (0, \ldots, 0, \underbrace{\zeta_j}_{j^{th}\text{ column}}, 0, \ldots, 0)' \in \mathcal{R}^{J\times 1}, \tag{4.23}$$

$$\mathbf{w}_j(\kappa) \equiv \begin{cases} (0, \ldots, 0, \underbrace{\phi_{Wj}(\kappa)}_{\kappa^{th}\text{ column}}, 0, \ldots, 0)', & \kappa \neq j \text{ and } \kappa \in \Pi_E, \\ -\mathbf{w}_j, & \kappa = j \text{ and } \kappa \in \Pi_E, \end{cases} \tag{4.24}$$

$$\mathbf{h}_j(\kappa, i) \equiv \begin{cases} (0, \ldots, 0, \underbrace{\phi_{Hj}(\kappa, i)}_{\kappa^{th}\text{column}}, 0, \ldots, 0)', & \kappa \neq j \text{ and } \kappa \in \Pi_E, \\ -\mathbf{h}_j(i), & \kappa = j \text{ and } \kappa \in \Pi_E, \end{cases} \tag{4.25}$$

$$\mathbf{g}_j(\kappa) \equiv \begin{cases} 0, & \kappa \neq j \text{ and } \kappa \in \Pi_E, \\ -\mathbf{g}_j, & \kappa = j \text{ and } \kappa \in \Pi_E, \end{cases} \tag{4.26}$$

$$\eta_j(\kappa) \equiv \begin{cases} 0, & \kappa \neq j, \text{ or } \kappa = j \in \Pi_G, \\ 1, & \kappa = j \in \Pi_E, \end{cases} \tag{4.27}$$

$$\theta_j(\kappa) \equiv \begin{cases} 0, & \kappa \neq j, \text{ or } \kappa = j \in \Pi_G, \\ \theta_j, & \kappa = j \in \Pi_E, \end{cases} \tag{4.28}$$

for $i = 1, \ldots, J$, where $'$ denotes a transposition of a vector or a matrix. Then the cost functions defined by (2.3), (2.4) and (2.5) of the system operated under a given scheduling algorithm are given by

$$W_j(\mathbf{Y}, e) = \begin{cases} r\varphi_j(\kappa) + g\psi_j(\kappa) + \mathbf{n}\mathbf{w}_j, & \kappa \in \Pi_G \\ r\varphi_j(\kappa) + g\psi_j(\kappa) + \mathbf{n}\mathbf{w}_j + \mathbf{n}\mathbf{w}_j(\kappa), & \kappa \in \Pi_E, \end{cases} \tag{4.29}$$

$$H_j(\mathbf{Y}, e, i) = \begin{cases} r\varphi_j(\kappa, i) + g\psi_j(\kappa, i) + \mathbf{n}\mathbf{h}_j(i), & \kappa \in \Pi_G, \\ r\varphi_j(\kappa, i) + g\psi_j(\kappa, i) + \mathbf{n}\mathbf{h}_j(i) + \mathbf{n}\mathbf{h}_j(\kappa, i), & \kappa \in \Pi_E, \end{cases} \tag{4.30}$$

$$G_j(\mathbf{Y}, e) = \begin{cases} r\eta_j(\kappa) + g\theta_j(\kappa) + \mathbf{n}\mathbf{g}_j, & \kappa \in \Pi_G, \\ r\eta_j(\kappa) + g\theta_j(\kappa) + \mathbf{n}\mathbf{g}_j + \mathbf{n}\mathbf{g}_j(\kappa), & \kappa \in \Pi_E, \end{cases} \tag{4.31}$$

for $k = j$ and $i = 1, \ldots, J$. For $k \neq j$,

$$W_j(\mathbf{Y}, e) = 0, \tag{4.32}$$

$$H_j(\mathbf{Y}, e, i) = 0, \tag{4.33}$$

$$G_j(\mathbf{Y}, e) = 0. \tag{4.34}$$

The important thing to consider about the above expressions is that the component $(k, \kappa, r, g, \mathbf{n})$ of state $\mathbf{Y}$ is sufficient to derive the expected values of these quantities. Further each function is linear with respect to $(r, g, \mathbf{n})$.

## 5. System at the steady state

In this section, we evaluate steady state values of the cost functions $W_j$, $H_j$ and $G_j$.

Now we define waiting times of the $e^{th}$ customer at station $j (j = 1, \ldots, J)$ as follows:

$$W_j^e \equiv \int_0^\infty C_{Wj}^e(t)dt, \tag{5.1}$$

$$H_j^e(i) \equiv \int_0^\infty C_{Wj}^e(t)1\{\kappa(t) = i\}dt, \tag{5.2}$$

$$G_j^e \equiv \int_0^\infty C_{Gj}^e(t)dt, \tag{5.3}$$

for $e = 1, 2, \ldots$. $W_j^e$ is the waiting time of the $e^{th}$ customer spent at station $j$ from his arrival to the beginning of his service period, and $H_j^e(i)$ is his waiting time at station $j$ spent during the system is in service periods of station $i$ until his service period begins. $G_j^e$ is his waiting time at station $j$ spent from when the server select station $j$ to serve him until his service is started. Now we define

$$\bar{w}_j^q \equiv \lim_{N \to \infty} \frac{\sum_{e=1}^N (W_j^e + G_j^e)}{\sum_{e=1}^N 1\{X(\sigma^e) = j\}}. \tag{5.4}$$

Since $W_j^e = 0$ and $G_j^e = 0$ for $X(\sigma^e) \neq j$, $\bar{w}_j^q$ denotes a steady state value of the mean waiting time that a class $j$ customer spends in the system from his arrival to the beginning of his service. To obtain these values, we define:

$$\bar{W}_j(\kappa) \equiv \lim_{N \to \infty} \frac{1}{N} \sum_{e=1}^N W_j^e 1\{\kappa(\sigma^e) = \kappa\}, \tag{5.5}$$

$$\bar{H}_j(\kappa,i) \equiv \lim_{N\to\infty}\frac{1}{N}\sum_{e=1}^{N} H_j^e(i)\mathbf{1}\{\kappa(\sigma^e)=\kappa\}, \tag{5.6}$$

$$\bar{G}_j(\kappa) \equiv \lim_{N\to\infty}\frac{1}{N}\sum_{e=1}^{N} G_j^e\mathbf{1}\{\kappa(\sigma^e)=\kappa\}, \tag{5.7}$$

for $\kappa \in \Pi$. Further we define

$$\bar{W}_j \equiv \sum_{\kappa=0}^{J}\bar{W}_j(\kappa) = \lim_{N\to\infty}\frac{1}{N}\sum_{e=1}^{N} W_j^e, \tag{5.8}$$

$$\bar{H}_j(i) \equiv \sum_{\kappa=0}^{J}\bar{H}_j(\kappa,i) = \lim_{N\to\infty}\frac{1}{N}\sum_{e=1}^{N} H_j^e(i), \tag{5.9}$$

$$\bar{G}_j \equiv \sum_{\kappa=0}^{J}\bar{G}_j(\kappa) = \lim_{N\to\infty}\frac{1}{N}\sum_{e=1}^{N} G_j^e. \tag{5.10}$$

We are willing to assume that

[A-1 ] the process $\mathcal{Q}$ is regenerative [6].

Let $N_B$ be the number of customers served during a regenerative cycle. Further we assume that

[A-2 ] the system is initially empty, and

[A-3 ] $E[N_B] < \infty$.

These assumptions are necessary to represent the above customer average values of the cost functions as follows:

$$\bar{W}_j(\kappa) = \frac{E[\sum_{e=1}^{N_B} W_j^e\mathbf{1}\{\kappa(\sigma^e)=\kappa\}]}{E[N_B]}, \tag{5.11}$$

$$\bar{H}_j(\kappa,i) = \frac{E[\sum_{e=1}^{N_B} H_j^e(i)\mathbf{1}\{\kappa(\sigma^e)=\kappa\}]}{E[N_B]}, \tag{5.12}$$

$$\bar{G}_j(\kappa) = \frac{E[\sum_{e=1}^{N_B} G_j^e\mathbf{1}\{\kappa(\sigma^e)=\kappa\}]}{E[N_B]},, \tag{5.13}$$

if we may assume that the numerators in the right-hand side of the above expressions are finite ($\kappa \in \Pi$). The customer average values $\bar{\mathbf{Y}}^\kappa \equiv (\bar{X}^\kappa, \kappa\bar{q}^\kappa, \bar{r}^\kappa, \bar{g}^\kappa, \bar{\mathbf{n}}^\kappa, \bar{L}^\kappa)$ of the state at customer's arrival epochs where $\bar{\mathbf{n}}^\kappa \equiv (\bar{n}_1^\kappa, \ldots, \bar{n}_J^\kappa)$ are defined by:

$$\bar{\mathbf{Y}}^\kappa \equiv \lim_{N\to\infty}\frac{1}{N}\sum_{e=1}^{N}\mathbf{Y}(\sigma^e)\mathbf{1}\{\kappa(\sigma^e)=\kappa\} \tag{5.14}$$

where $\bar{q}^\kappa$ is a fraction of arrivals who find a class $\kappa$ customer being served. Then we assume that the customer average values exist and that

$$[\text{A-4}] \quad \begin{cases} E[\sum_{e=1}^{N_B} r(\sigma^e) 1\{\kappa(\sigma^e) = \kappa\}] < \infty, \\ E[\sum_{e=1}^{N_B} g(\sigma^e) 1\{\kappa(\sigma^e) = \kappa\}] < \infty, \\ E[\sum_{e=1}^{N_B} \mathbf{n}(\sigma^e) 1\{\kappa(\sigma^e) = \kappa\}] < \infty, \end{cases}$$

Then we have

$$\bar{r}^\kappa = \frac{E[\sum_{e=1}^{N_B} r(\sigma^e) 1\{\kappa(\sigma^e) = \kappa\}]}{E[N_B]} < \infty, \tag{5.15}$$

$$\bar{g}^\kappa = \frac{E[\sum_{e=1}^{N_B} g(\sigma^e) 1\{\kappa(\sigma^e) = \kappa\}]}{E[N_B]} < \infty, \tag{5.16}$$

$$\bar{\mathbf{n}}^\kappa = \frac{E[\sum_{e=1}^{N_B} \mathbf{n}(\sigma^e) 1\{\kappa(\sigma^e) = \kappa\}]}{E[N_B]} < \infty. \tag{5.17}$$

Further the customer average values $\bar{\mathbf{Y}} \equiv (\bar{X}, \bar{\kappa}, \bar{r}, \bar{g}, \bar{\mathbf{n}}, \bar{L})$ of the state are defined by

$$\bar{\mathbf{Y}} \equiv \sum_{\kappa=0}^{J} \bar{\mathbf{Y}}^\kappa = \lim_{N\to\infty} \frac{1}{N} \sum_{e=1}^{N} \mathbf{Y}(\sigma^e), \tag{5.18}$$

The time average values $\tilde{\mathbf{Y}}^\kappa \equiv (\tilde{X}^\kappa, \kappa\tilde{q}^\kappa, \tilde{r}^\kappa, \tilde{g}^\kappa, \tilde{\mathbf{n}}^\kappa, \tilde{L}^\kappa)$, and $\tilde{\mathbf{Y}} \equiv (\tilde{X}, \tilde{\kappa}, \tilde{r}, \tilde{g}, \tilde{\mathbf{n}}, \tilde{L})$ of the state are defined by:

$$\tilde{\mathbf{Y}}^\kappa \equiv \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{Y}(s) 1\{\kappa(s) = \kappa\} ds, \quad \kappa = 0, 1, \ldots, J, \tag{5.19}$$

$$\tilde{\mathbf{Y}} \equiv \sum_{\kappa=0}^{J} \tilde{\mathbf{Y}}^\kappa = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{Y}(s) ds, \tag{5.20}$$

where $\tilde{q}^\kappa$ is a fraction of time that a class $\kappa$ customer is being served.

Now we get the following representations concerned with the steady state values of the cost functions. By (4.29), (4.30), (4.31), (5.15), (5.16) and (5.17), the numerators of the right-hand side of equations (5.11), (5.12) and (5.13) are shown to be finite. Hence, from (4.29), (4.30) and (4.31), and from (5.11) through (5.17), we can show that

$$\bar{W}_j(\kappa) = \begin{cases} (\lambda_j/\lambda)\{\bar{r}^\kappa \varphi_j(\kappa) + \bar{g}^\kappa \psi_j(\kappa) + \bar{\mathbf{n}}^\kappa \mathbf{w}_j\}, & \kappa \in \Pi_G \\ (\lambda_j/\lambda)\{\bar{r}^\kappa \varphi_j(\kappa) + \bar{g}^\kappa \psi_j(\kappa) + \bar{\mathbf{n}}^\kappa \mathbf{w}_j + \bar{\mathbf{n}}^\kappa \mathbf{w}_j(\kappa)\}, & \kappa \in \Pi_E, \end{cases} \tag{5.21}$$

$$\bar{H}_j(\kappa, i) = \begin{cases} (\lambda_j/\lambda)\{\bar{r}^\kappa \varphi_j(\kappa, i) + \bar{g}^\kappa \psi_j(\kappa, i) + \bar{\mathbf{n}}^\kappa \mathbf{h}_j(i)\}, & \kappa \in \Pi_G, \\ (\lambda_j/\lambda)\{\bar{r}^\kappa \varphi_j(\kappa, i) + \bar{g}^\kappa \psi_j(\kappa, i) + \bar{\mathbf{n}}^\kappa \mathbf{h}_j(i) + \bar{\mathbf{n}}^\kappa \mathbf{h}_j(\kappa, i)\}, & \kappa \in \Pi_E, \end{cases} \tag{5.22}$$

$$\bar{G}_j(\kappa) = \begin{cases} (\lambda_j/\lambda)\{\bar{r}^\kappa\eta_j(\kappa) + \bar{g}^\kappa\theta_j(\kappa) + \bar{\mathbf{n}}^\kappa \mathbf{g}_j\}, & \kappa \in \Pi_G, \\ (\lambda_j/\lambda)\{\bar{r}^\kappa\eta_j(\kappa) + \bar{g}^\kappa\theta_j(\kappa) + \bar{\mathbf{n}}^\kappa \mathbf{g}_j + \bar{\mathbf{n}}^\kappa \mathbf{g}_j(\kappa)\}, & \kappa \in \Pi_E. \end{cases} \tag{5.23}$$

The steady state value $\tilde{r}^\kappa$ of a remaining service time is given by

$$\tilde{r}^\kappa = \frac{\lambda_\kappa E[S_\kappa^2]}{2}, \quad \kappa = 1, \ldots, J. \tag{5.24}$$

It is obvious that $\tilde{r}^0 = 0$. For the Poisson arrival, the fraction of time that the system is in any state is equal to the fraction of the arrivals when the system is in the state. This is the PASTA property [11]. Thus $\tilde{r}^\kappa$ is equivalent to $\bar{r}^\kappa$. We also use the generalized Little's formula ($H = \lambda G$) [9] that equates the time average values of the costs with the customer average values of the costs to obtain

$$\tilde{n}_j = \lambda \bar{W}_j, \tag{5.25}$$

$$\tilde{n}_j^i = \lambda \bar{H}_j(i), \tag{5.26}$$

$$\tilde{g}^j = \lambda \bar{G}_j. \tag{5.27}$$

Obviously, we have $\tilde{n}_j^0 = 0$ $(j = 1, \ldots, J)$ and $\tilde{g}^0 = 0$. As a matter of convenience, we only consider the case of $\kappa \in \Pi_E$. Then from (5.8), (5.18), (5.21) and (5.25), we have

$$\tilde{n}_j = \lambda_j \left\{ \sum_{\kappa=0}^{J} \bar{r}^\kappa \varphi_j(\kappa) + \sum_{\kappa=0}^{J} \bar{g}^\kappa \psi_j(\kappa) + \bar{\mathbf{n}}\mathbf{w}_j + \sum_{\kappa \in \Pi_E} \bar{\mathbf{n}}^\kappa \mathbf{w}_j(\kappa) \right\} \tag{5.28}$$

From (5.9), (5.18), (5.22) and (5.26), we have

$$\tilde{n}_j^i = \lambda_j \left\{ \sum_{\kappa=0}^{J} \bar{r}^\kappa \varphi_j(\kappa, i) + \sum_{\kappa=0}^{J} \bar{g}^\kappa \psi_j(\kappa, i) + \bar{\mathbf{n}}\mathbf{h}_j(i) + \sum_{\kappa \in \Pi_E} \bar{\mathbf{n}}^\kappa \mathbf{h}_j(\kappa, i) \right\} \tag{5.29}$$

From (5.10), (5.18), (5.23) and (5.27), we have

$$\tilde{g}^j = \lambda_j \left\{ \sum_{\kappa=0}^{J} \bar{r}^\kappa \eta_j(\kappa) + \sum_{\kappa=0}^{J} \bar{g}^\kappa \theta_j(\kappa) + \bar{\mathbf{n}}\mathbf{g}_j + \sum_{\kappa \in \Pi_E} \bar{\mathbf{n}}^\kappa \mathbf{g}_j(\kappa) \right\} \tag{5.30}$$

From the PASTA property, we have

$$\tilde{n}_j = \lambda_j \left\{ \sum_{\kappa=1}^{J} \tilde{r}^\kappa \varphi_j(\kappa) + \sum_{\kappa=1}^{J} \tilde{g}^\kappa \psi_j(\kappa) + \tilde{\mathbf{n}}\mathbf{w}_j + \sum_{\kappa \in \Pi_E} \tilde{\mathbf{n}}^\kappa \mathbf{w}_j(\kappa) \right\}, \tag{5.31}$$

$$\tilde{n}_j^i = \lambda_j \left\{ \sum_{\kappa=1}^{J} \tilde{r}^\kappa \varphi_j(\kappa, i) + \sum_{\kappa=1}^{J} \tilde{g}^\kappa \psi_j(\kappa, i) + \tilde{\mathbf{n}}\mathbf{h}_j(i) + \sum_{\kappa \in \Pi_E} \tilde{\mathbf{n}}^\kappa \mathbf{h}_j(\kappa, i) \right\}, \tag{5.32}$$

$$\tilde{g}^j = \lambda_j \left\{ \sum_{\kappa=1}^{J} \tilde{r}^\kappa \eta_j(\kappa) + \sum_{\kappa=1}^{J} \tilde{g}^\kappa \theta_j(\kappa) + \tilde{\mathbf{n}}\mathbf{g}_j + \sum_{\kappa \in \Pi_E} \tilde{\mathbf{n}}^\kappa \mathbf{g}_j(\kappa) \right\} \tag{5.33}$$

For the simplicity of the expression of above equations, we define the following vectors and matrices.

$$\tilde{\mathbf{g}} = (\tilde{g}^1, \ldots, \tilde{g}^J),$$

$$\mathbf{s}_g = \left(\sum_{\kappa=1}^{J} \tilde{r}^\kappa \eta_1(\kappa), \ldots, \sum_{\kappa=1}^{J} \tilde{r}^\kappa \eta_J(\kappa)\right), \quad \mathbf{s}_w = \left(\sum_{\kappa=1}^{J} \tilde{r}^\kappa \varphi_1(\kappa), \ldots, \sum_{\kappa=1}^{J} \tilde{r}^\kappa \varphi_J(\kappa)\right),$$

$$\mathbf{s}_w(i) = \left(\sum_{\kappa=1}^{J} \tilde{r}^\kappa \varphi_1(\kappa, i), \ldots, \sum_{\kappa=1}^{J} \tilde{r}^\kappa \varphi_J(\kappa, i)\right),$$

$$\Theta = \begin{pmatrix} \theta_1(1) & \theta_2(1) & \cdots & \theta_J(1) \\ \theta_1(2) & \theta_2(2) & \cdots & \theta_J(2) \\ \vdots & \vdots & \ddots & \vdots \\ \theta_1(J) & \theta_2(J) & \cdots & \theta_J(J) \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1(1) & \psi_2(1) & \cdots & \psi_J(1) \\ \psi_1(2) & \psi_2(2) & \cdots & \psi_J(2) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(J) & \psi_2(J) & \cdots & \psi_J(J) \end{pmatrix},$$

$$\Psi(i) = \begin{pmatrix} \psi_1(1,i) & \psi_2(1,i) & \cdots & \psi_J(1,i) \\ \psi_1(2,i) & \psi_2(2,i) & \cdots & \psi_J(2,i) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(J,i) & \psi_2(J,i) & \cdots & \psi_J(J,i) \end{pmatrix},$$

$$\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_J), \quad \mathbf{g}(\kappa) = (\mathbf{g}_1(\kappa), \ldots, \mathbf{g}_J(\kappa)),$$

$$\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_J), \quad \mathbf{w}(\kappa) = (\mathbf{w}_1(\kappa), \ldots, \mathbf{w}_J(\kappa)),$$

$$\mathbf{h}(i) = (\mathbf{h}_1(i), \ldots, \mathbf{h}_J(i)), \quad \mathbf{h}(\kappa, i) = (\mathbf{h}_1(\kappa, i), \ldots, \mathbf{h}_J(\kappa, i)),$$

$$\boldsymbol{\Lambda} = diag\{\lambda_j : j = 1, \ldots, J\}.$$

Then we have

$$\tilde{\mathbf{g}} = \left\{ \mathbf{s}_g + \tilde{\mathbf{g}}\Theta + \tilde{\mathbf{n}}\mathbf{g} + \sum_{\kappa \in \Pi_E} \tilde{\mathbf{n}}^\kappa \mathbf{g}(\kappa) \right\} \boldsymbol{\Lambda}, \tag{5.34}$$

$$\tilde{\mathbf{n}} = \left\{ \mathbf{s}_w + \tilde{\mathbf{g}}\Psi + \tilde{\mathbf{n}}\mathbf{w} + \sum_{\kappa \in \Pi_E} \tilde{\mathbf{n}}^\kappa \mathbf{w}(\kappa) \right\} \boldsymbol{\Lambda}, \tag{5.35}$$

$$\tilde{\mathbf{n}}^i = \left\{ \mathbf{s}_w(i) + \tilde{\mathbf{g}}\Psi(i) + \tilde{\mathbf{n}}\mathbf{h}(i) + \sum_{\kappa \in \Pi_E} \tilde{\mathbf{n}}^\kappa \mathbf{h}(\kappa, i) \right\} \boldsymbol{\Lambda}. \tag{5.36}$$

Let $J_E$ be the number of the stations with exhaustive disciplines and let $i_1, \ldots, i_{J_E}$ denote the stations with exhaustive disciplines. Then $\Pi_E = \{i_1, \ldots, i_{J_E}\}$. Further let $J^* = (2 + J_E) \times J$ be the number of the unknowns $\tilde{\mathbf{g}}, \tilde{\mathbf{n}}, \tilde{\mathbf{n}}^i$ $(i = i_1, \ldots, i_{J_E})$. We define vectors and matrices:

$$\tilde{\mathbf{y}} = (\tilde{\mathbf{g}}, \tilde{\mathbf{n}}, \tilde{\mathbf{n}}^{i_1}, \ldots, \tilde{\mathbf{n}}^{i_{J_E}}) \in \mathcal{R}^{1 \times J^*},$$

$$\mathbf{s} = (\mathbf{s}_g, \mathbf{s}_w, \mathbf{s}_w(i_1), \ldots, \mathbf{s}_w(i_{J_E})) \in \mathcal{R}^{1 \times J^*},$$

$$\mathbf{S} = \begin{pmatrix} \Theta & \Psi & \Psi(i_1) & \cdots & \Psi(i_{J_E}) \\ \mathbf{g} & \mathbf{w} & \mathbf{h}(i_1) & \cdots & \mathbf{h}(i_{J_E}) \\ \mathbf{g}(i_1) & \mathbf{w}(i_1) & \mathbf{h}(i_1, i_1) & \cdots & \mathbf{h}(i_1, i_{J_E}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{g}(i_{J_E}) & \mathbf{w}(i_{J_E}) & \mathbf{h}(i_{J_E}, i_1) & \cdots & \mathbf{h}(i_{J_E}, i_{J_E}) \end{pmatrix} \in \mathcal{R}^{J^* \times J^*},$$

$$\boldsymbol{\Lambda}_* = diag\{\lambda_1, \ldots, \lambda_J, \lambda_1, \ldots, \lambda_J, \ldots, \lambda_1, \ldots, \lambda_J\} \in \mathcal{R}^{J^* \times J^*}.$$

Then we arrive at an equation that determines a steady state value of the components of the process:

$$\tilde{\mathbf{y}} = \{\mathbf{s} + \tilde{\mathbf{y}}\mathbf{S}\} \boldsymbol{\Lambda}_*. \tag{5.37}$$

If we assume that the inverse matrices exist, we have

$$\tilde{\mathbf{y}} = \mathbf{s} \left( \boldsymbol{\Lambda}_*^{-1} - \mathbf{S} \right)^{-1} \tag{5.38}$$

Finally, we can get steady state values $\tilde{\mathbf{n}}^q \equiv (\tilde{n}_1^q, \ldots, \tilde{n}_J^q)$ of the number of customers in the system, and steady state values $\tilde{\mathbf{w}}^q \equiv (\tilde{w}_1^q, \ldots, \tilde{w}_J^q)$ of the mean waiting times from the arrival of a customer to the beginning of his service in every class:

$$\tilde{\mathbf{n}}^q = \tilde{\mathbf{n}} + \tilde{\mathbf{g}}, \tag{5.39}$$

$$\tilde{\mathbf{w}}^q = \tilde{\mathbf{n}}^q \boldsymbol{\Lambda}^{-1}. \tag{5.40}$$

## 6. Numerical examples

For numerical examples, we consider a system with $J = 8$ classes of customers. We assume that the parameters of the arrival and service processes are given by

1. The arrival rate of each class of customers.
   - $\lambda_j = 1/40.0 : (j = 1, 2)$,
   - $\lambda_j = 1/45.0 : (j = 3, 4)$,
   - $\lambda_j = 1/50.0 : (j = 5, 6)$,
   - $\lambda_j = 1/55.0 : (j = 7, 8)$.
2. The service time distributions are the 5 stage Erlang distributions. Their mean values increase by 0.2 from the following initial values:
   - $E[S_j] = 0.40 : (j = 1, 2)$,
   - $E[S_j] = 0.50 : (j = 3, 4)$,
   - $E[S_j] = 0.60 : (j = 5, 6)$,
   - $E[S_j] = 0.70 : (j = 7, 8)$.

The mean waiting times $\bar{w}_j^q (j = 1, \ldots, 8)$ for each class of customers have been computed, and plotted against $\rho$. In Figure 1, we considered a model which all stations adopt exhaustive disciplines. In Figure 2, we considered a model which the stations from 1 through 4 adopt exhaustive disciplines, and from 5 through 8 adopt gated disciplines. In Figure 3, we considered a model which the stations from 1 through 4 adopt gated disciplines, and from 5 through 8 adopt exhaustive disciplines.

## 7. Conclusion

We have investigated a new approach to the analysis of multiclass M/G/1 system with priority. We have treated priority scheduling algorithms where each station adopts either gated discipline or exhaustive discipline. We first have defined states of the system and a stochas-

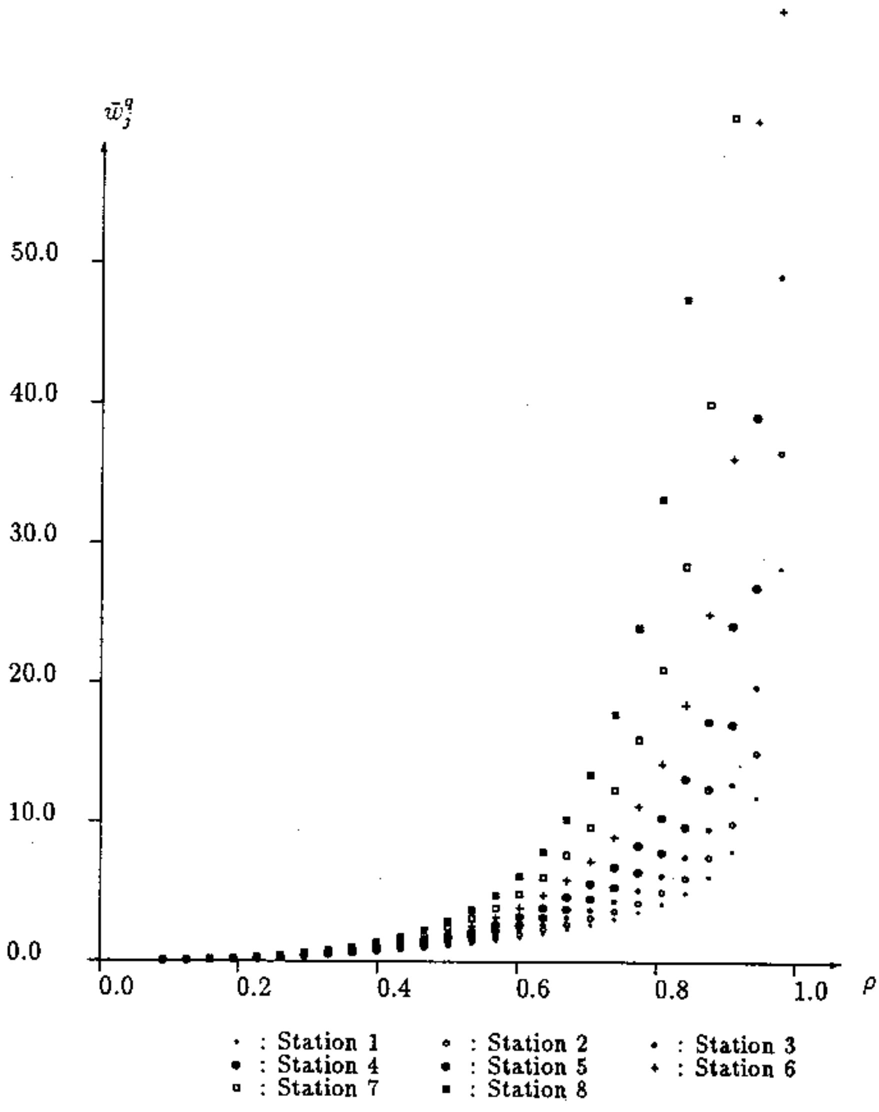


Figure 1: All stations adopt exhaustive discipline

tic process which represents an evolution of the system, and then the system performance measures as cost functions of the states. Second, we have derived expressions of these cost functions for every station and every service discipline. The important things to consider about these expressions are that the component $(k, \kappa, r, g, \mathbf{n})$ of state $\mathbf{Y} \in \mathcal{E}$ is sufficient to derive them and that each function is linear with respect to $(r, g, \mathbf{n})$ for any given $k$ and

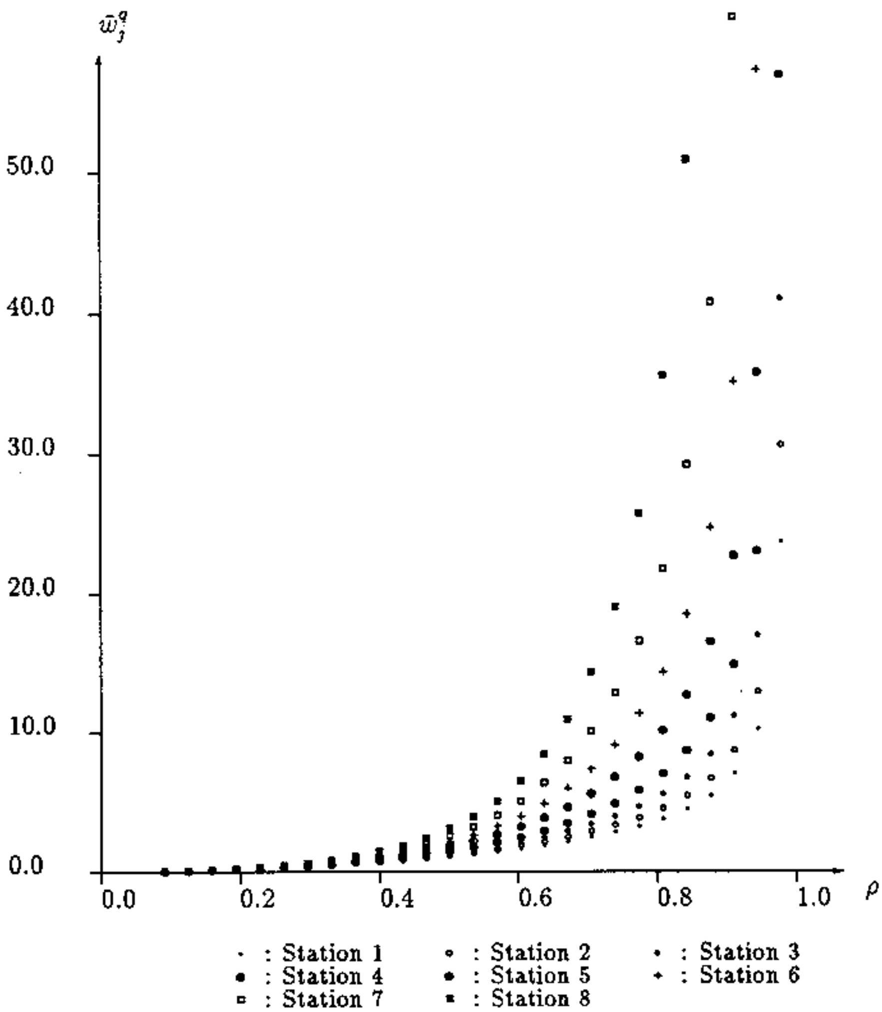


Figure 2: The stations from 1 through 4 adopt exhaustive discipline, and from 5 through 8 adopt gated discipline.

$\kappa$. Hence the system performance measures can be considered to be output processes of the process $\mathcal{Q}$. Finally, we have evaluated their steady state values by using Poisson arrivals see time averages and the generalized Little's formula. As we can obtain $J^* = (2 + J_E) \times J$ equations for $J^*$ unknowns, we can solve them. Since these solutions are expressed in matrix forms, an algorithm for yielding their actual values can be easily constructed.

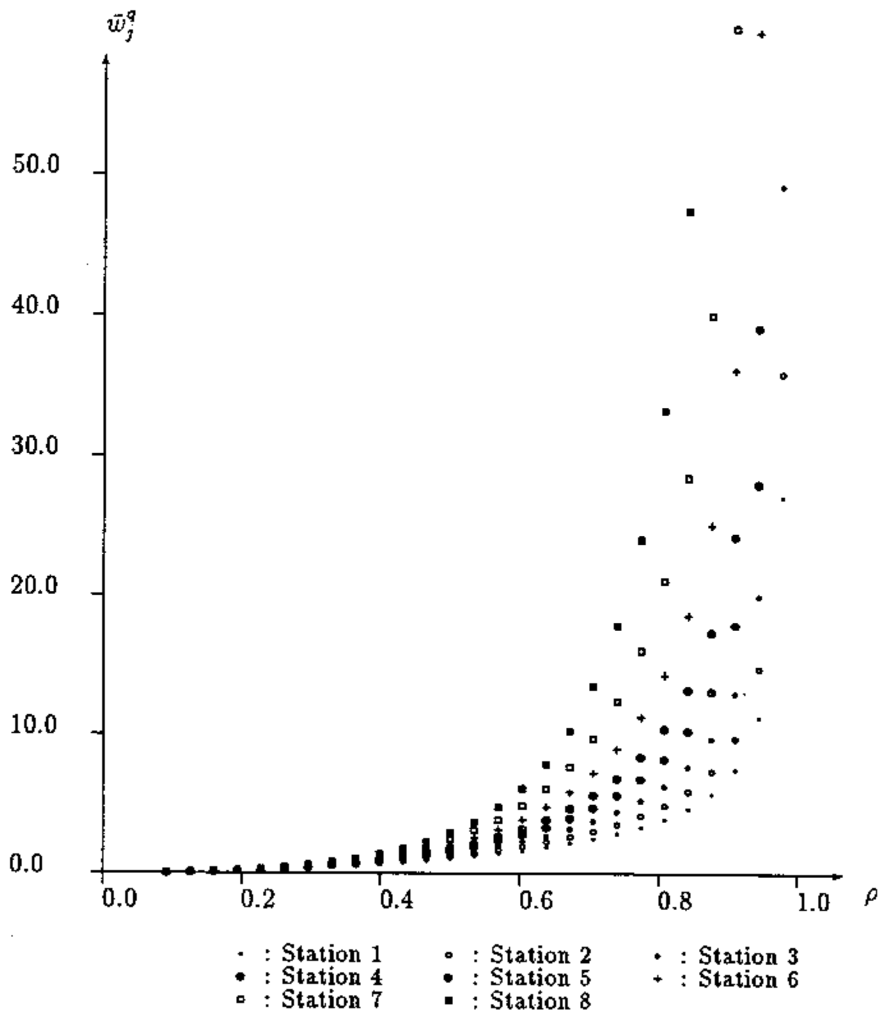


Figure 3: The stations from 1 through 4 adopt gated discipline, and from 5 through 8 adopt exhaustive discipline.

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