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외곽선이 Smooth한 객체의 Medial 축 변환에의 새로운 접근 방법

A New Approach to Medial Axis Transformation of Objects with Smooth Boundary

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Abstract

Medial axis transformation is an important concept used in many engineering applications. We propose a new approach to medial axis transformation of 2D objects with smooth boundary. Our approach differs from the traditional ones: we construct the medial axis starting from the inside points, while the previous algorithms started from the boundary points. As a result, previous algorithms are highly sensitive to the small irregularities of the object's boundary curve, while our approach is robust.

1. Introduction

There are various schemes to represent a geometric object. Among them, an important and useful approach to extract the structural shape of an object is to reduce it to a graph. An important concept used for this reduction is the medial axis (skeleton) of the object. Medial axis was first proposed by Blum[3]. Intuitively the medial axis of a 2-dimensional object R with boundary B is as follows. For each point p in R, we find its closest neighbor on B. If p has more than one such neighbor, it is said to belong to the medial axis of R. Naturally the concept of "closest" depends on the definition of a distance. In this paper we use the Euclidean distance. Therefore each point on the medial axis is a center of a maximal-sized disk that is contained within the shape.

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Medial axis transformation are used in many engineering applications. Medial axis transformation extracts the most essential structural features from the image data and represent the image in a simple graph with nodes and edges. This simple representation of the image is valuable in problems such as identifying the objects in pattern recognition and storing the image data in computer graphics. Medial axis transformation is also necessary to find the offset curves in font design, rapid prototyping, solid modeling and NC machining. In fact, our interest in the medial axis transformation came from our work in designing Korean and Chinese characters.

Although the medial axis of a plane object yields a good characteristics for the structural shape, direct implementation of that definition has been known computationally prohibitive because of the fact that implementation involves calculating the distance from every interior point to every point on the boundary of the object. Numerous algorithms have been proposed to produce a medial axis for digital shapes. Montanari[4] describes an algorithm for finding the medial axis of a shape whose boundary can be piecewise approximated by straight-line segments and circular arcs. Typically these algorithms are thinning algorithms that iteratively delete edge points of a digitized region[5][6]. Also Lantuejoule[7] characterized medial axis transformation in terms of morphological operations for digital shapes[8]. These previous works constructed the medial axis from the boundary points either by thinning algorithms or by emnating wave front perpendicularly from the boundary points. Therefore their result was highly sensitive to small irregularities of the object's boundary curve. However, our approach differs from the traditional one: we construct the medial axis starting from the inside points not from the boundary points. We construct a series of inscribed circle. Then we reconstruct the envelope of these circles. If this envelope is close enough to the given object's boundary curve, that series of inscribed circles comprise the medial axis transformation of the given object. Our approach, starting from the inside, produces a robust result to the small irregularities of the object's boundary curve. We are currently studying this stability properties of our approach in more details. In this paper we assume that the objects have smooth boundary. We are also working on the medial axis transformation for objects with more general shape.

In Section 2 we state the formal definitions of medial axis transformation. Our definition is slightly different from the traditional ones in the sense that it searches the medial axis from the inside of the object and our definitions also include the center point of the inscribed osculating circles. In Section 3 we review the problem of sweeping a 2-dimensional object of a fixed shape with algebraic boundary curves. As it turns out, we can get explicit formula for the envelopes in the case of sweeping circles. In Section 4 we describe how to construct the medial axis transformation of objects with smooth boundary. We start with a finite number of points on the medial axis. We reconstruct a sweep operation which passes through these data points. Our procedure repeats these steps until the envelope of the reconstructed sweep is close enough to the boundary of the given object. Finally, in Section 5 we make concluding remarks and discuss further study.

2. Medial Axis Tansformation

Let Q be a bounded closed domain in R^2 with piecewise $C^k(k \geq 1)$ boundary which has only finitely many non-differentiable points. Let $B_r(p)$ denote the closed disk of radius r centered at p. We define the ordered set $D(Q)$ by

$$D(Q) = \{B_r(p) \mid B_r(p) \subset Q\} \qquad \text{(Eq. 2.1)}$$

That is, $D(Q)$ is the set of all disks contained in Q.

Now we define the medial axis and the medial axis transformation. Our definition is slightly different from the original definition by Blum[3].

DEFINITION 2.1. The *CORE* of a domain Q is the set of all maximal disks in Q, that is,

$$CORE(Q) = \{B_r(p) \in D(Q) \mid \forall B_s(q) \in D(Q), B_r(p) \subset B_s(q) \mathit{implies} B_r(p) = B_s(q)\}$$

DEFINITION 2.2. The *MEDIAL AXIS* of a domain Q is the set of all centers of disks in $CORE(Q)$. That is,

$$MA(Q) = \{p \in Q \mid B_r(p) \in CORE(Q)\}$$

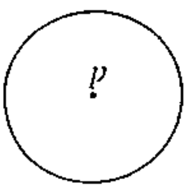
DEFINITION 2.3. The *MEDIAL AXIS TRANSFORMATION* of a domain Q is the set of all ordered pairs of cneters and radii of disks in $CORE(Q)$. That is,

$$MAT(Q) = \{(p,r) \in Q \times R^+ \cup \{0\} \mid B_r(p) \in CORE(Q)\}$$

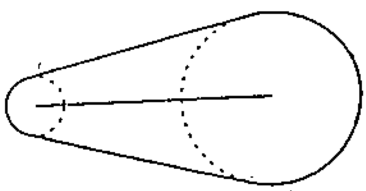
Now let us explain the geometric meaning of the medial axis. Let $d(p,q)$ denote the distance between points p and q. In this paper we use Euclidean distance. By using different distance function, one get different medial axis. The distance from a point $p \in Q$ to the boundary ∂Q will be denoted by $d(p, \partial Q)$, that is, $d(p, \partial Q) = \min_{q \in \partial Q} d(p,q)$.

Traditionally, the medial axis is defined to be the set of point at which wave front emnating perpendicularly from two or more boundary points meet. Our definition of medial axis differs from the traditional one: we construct the medial axis starting from the inside points, while the traditional methods start from the boundary and our definitions also include the center point of the inscribed osculating circles. Direct implementation of the traditional definition of medial axis has been known computationally prohibitive because of the fact that implementation involves calculating the distance from every interior point to every point on the boundary of the object. Previous algorithms for constructing the medial axis of an object are typically thinning algorithms that iteratively delete edge points of a digitized region[5][6]. Also medial axis transformation was characterized in terms of morphological operations. However, these approaches are highly sensitive to small irregularities of the object's boundary curve, while our approach is quite robust to the small pertubations of the object's boundary curve. In this paper we assume that the objects have smooth boundary. We are currently investigating the medial axis transformation for objects with more general shapes. We are also studying the stability properties of our approach. Refer to our forthcoming papers.

In the rest of the paper, for a medial axis point p, let $B(p)$ denote the corresponding disk $B_r(p)$ in $CORE(Q)$. The distance $d(p, \partial Q)$ is realized at the contact point of $\partial B(p)$ with ∂Q. That is, for any $q \in \partial Q \cap \partial B(p)$ $d(p, \partial Q) = d(p,q)$. If the boundary ∂Q has a circular portion and p is the center of the circular portion, the distance $d(p, \partial Q)$ is realized at every point in the circular portion. Thus the distance can be realized at infintely many boundary points on the contcact arcs.

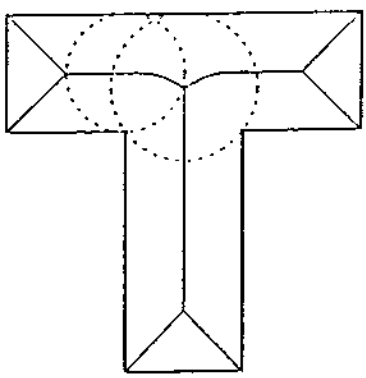


(a) circle



(b) stadium

(c) rectangle



(d) letter T

Figure 2.1 shows the medial axis of some simple 2-dimensional objects.

3. Envelope of a Family of Curves

Now we review the sweep operation which is widely used in solid modeling. The sweep operation is to generate a new solid by sweeping a given object along a given space curve trajectory. The simplest sweep is linear extrusion defined by sweeping a plane object along its normal direction. Rotational sweep is defined by rotating a 2-dimensional object about an axis. Sweeps whose generating area change in size, shape, or orientation as they are swept along a given trajectory are called general sweeps.[9]

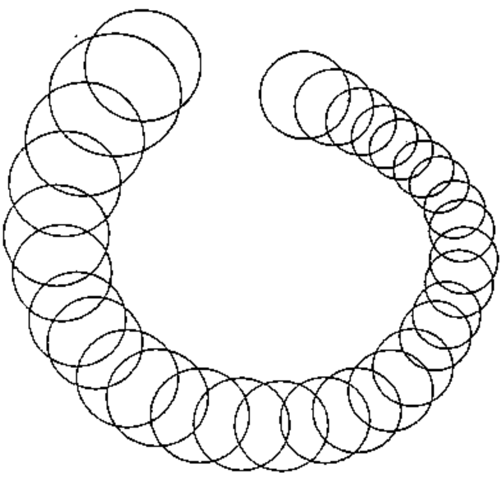


Figure 3.1 Sweeping dynamically changing circles

Consider the case of sweeping an object of a fixed shape with algebraic boundary curves. Let $F_t(x_1, x_2, \cdots\cdots, x_n) = 0$ be the boundary equation of the object at time t. The boundary of the swept area is called the *ENVELOPE* of a family of curves, $\{F_t\}$, and constructing the *ENVELOPE* is an important problem. The simplest case is sweeping a constant-radius 2-dimensional circle along a plane trajectory. A slightly more general case is sweeping a circle whose radius dynamically changes with time along a plane trajectory. Next Theorem states that for these simple cases of sweeps, the exact equation for the envelope can be found by solving simultaneous algebraic equations.

THEOREM: Consider the case of sweeping a 2-dimensional object of a fixed shape with algebraic boundary curves. Let $F(x,y,t) = 0$ be the boundary equation of the object at time t. Then the *ENVELOPE* of $F(x,y,t)$ is the set of all (x,y) which satisfies the following simultaneous equations:

$$F(x,y,t) = 0,$$

$$\frac{\partial F}{\partial t}(x,y,t) = 0 \qquad \text{(Eq. 3.1)}$$

Proof:

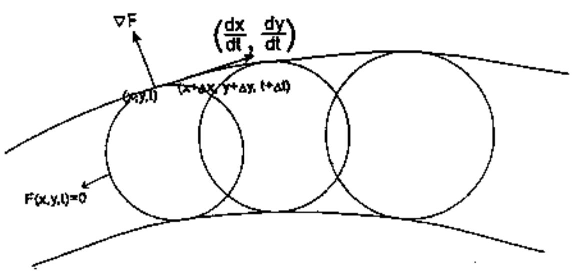


Figure 3.2 Finding envelope of family of circles

Let (x,y,t) and $(x+\Delta x, y+\Delta y, t+\Delta t)$ be two points on the envelope infinitesimally close to each other. Then $F(x,y,t) = 0$ and

$F(x+\Delta x, y+\Delta y, t+\Delta t)=0.$

Now $F(x+\Delta x, y+\Delta y, t+\Delta t)=F(x,y,t)+$

$$\frac{\partial F}{\partial x}\Delta x+\frac{\partial F}{\partial y}\Delta y+\frac{\partial F}{\partial t}\Delta t+O((\Delta t)^2)=0$$

By letting $F(x,y,t)=0$, dividing each side by Δt, and letting $\Delta t \rightarrow 0$, we get

$$\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial t}=0.$$

Since $(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y})$ is the gradient vector of the boundary curve, $\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}=0.$

Thus we get $\frac{\partial F}{\partial t}=0.$ Q.E.D.

Equation 3.1 gives a 2 simultaneous equations in terms of x,y,t. By eliminating the variable t, we can get closed-form algebraic equations in terms of x and y representing the sweep boundary. Although in case of sweeping a general object, eliminating t is computationally expensive, in case of sweeping a circle we can explicitly solve the simultaneous equations. Now consider the case of sweeping a 2-dimensional circle whose radius, $r(t)$, changes dynamically as a function of time t and whose center moves along the trajectory $(a(t),b(t))$ on the same plane.

In this case $F(x,y,t)=(x-a(t))^2+(y-b(t))^2-r(t)^2=0.$

So any point (x,y) on the sweep boundary satisfies the following equations:

$$F(x,y,t)=(x-a(t))^2+(y-b(t))^2-r(t)^2=0.$$

$$\frac{\partial F}{\partial t}=-2a'(t)(x-a(t))-2b'(t)(y-b(t))-2r'(t)r(t)=0$$

Eliminating t, we get the following equation for the sweep boundary:

$$x(t)=a(t)+(-a'(t)r(t)r'(t)\pm r(t)b'(t)(a'^2(t)+b'^2(t)-r'^2(t))^{\frac{1}{2}})/(a'^2(t)+b'^2(t))$$

$$y(t)=b(t)+(-b'(t)r(t)r'(t)\pm r(t)a'(t)(a'^2(t)+b'^2(t)-r'^2(t))^{\frac{1}{2}})/(a'^2(t)+b'^2(t)) \quad \text{(Eq. 3.2)}$$

4. Construction of Medial Axis

In this section, we propose a new approach for constructing the medial axis of a 2D object with smooth boundary. A number of algo-

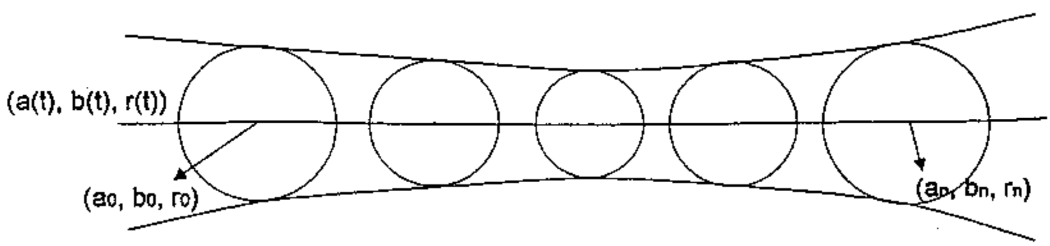


Figure. 4.1 Fitting (n+1) data points by spline functions

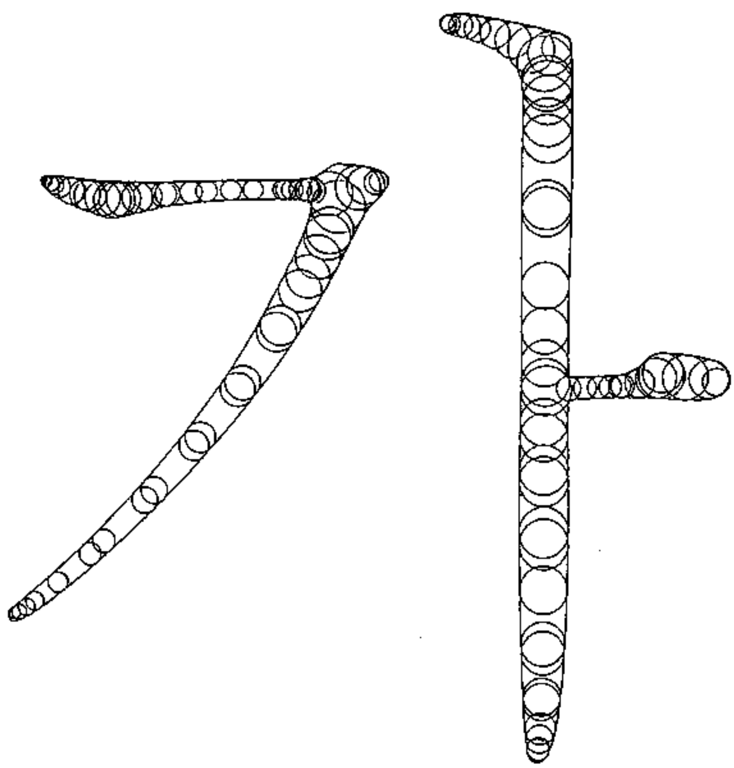
rithms are available for finding out a point on the medial axis. So we assume that we have found a finite number of points on the medial axis to start with. Let $\{p_0(a_0, b_0), p_1(a_1, b_1), \cdots, p_n(a_n, b_n)\}$ be the $n+1$ points on the medial axis and $\{r_0, r_1, \cdots, r_n\}$ be the radii of their corresponding maximal disks. Denote the maximal disk centered at $p_i(a_i, b_i)$ by $B(p_i)$. See Figure 4.1.

STEP 1. Given $n+1$ data points on the medial axis, $\{(a_0, b_0, r_0), (a_1, b_1, r_1), \cdots, (a_n, b_n, r_n)\}$,

1.1 compute the value of t for each data point as follows, which is known as chord length parametrization technique:

$t_0 = 0$, $t_i = t_{i-1} + d(p_i, p_{i-1})$ $(i = 1, \cdots, n)$,

where $d(p_{i-1}, p_i)$ is a distance between two points $p_{i-1}(a_{i-1}, b_{i-1})$ and $p_i(a_i, b_i)$

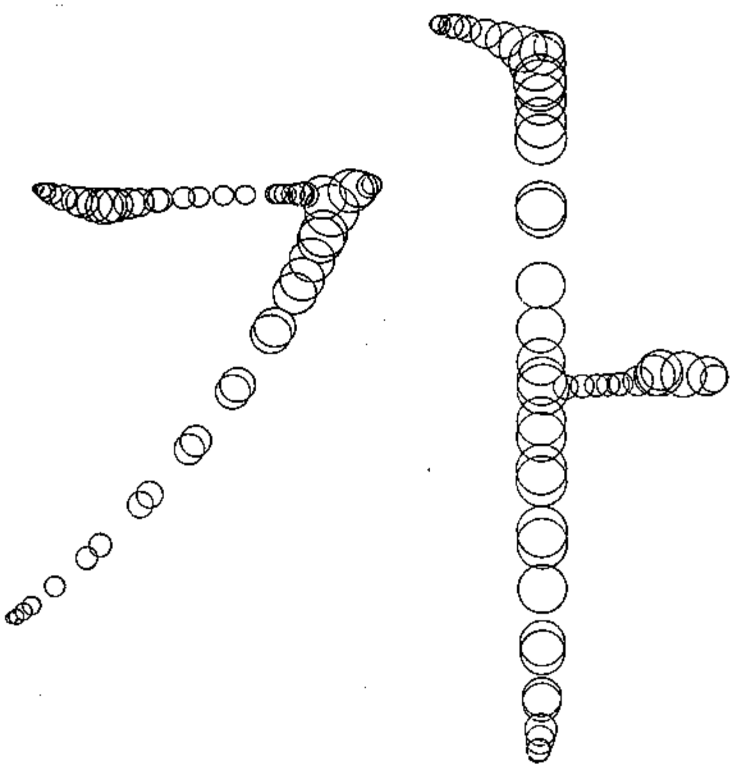


(a) Finding a number of medial axis points

Figure. 4.2 Medial axis transformation for Korean character 가

Now we propose the following method to construct the medial axis transformation:

1.2 construct a natural cubic spline function $a(t)$ that fits $\{(a_0, t_0), (a_1, t_1), \cdots, (a_n, t_n)\}$



(b) Reconstructing 가 from medial axis points

Figure. 4.2 Medial axis transformation for Korean character 가

1.3 construct a natural cubic spline function $b(t)$ that fits $\{(b_0,t_0),(b_1,t_1),\cdots,(b_n,t_n)\}$

1.4 construct a natural cubic spline function $r(t)$ that fits $\{(r_0,t_0),(r_1,t_1),\cdots,(r_n,t_n)\}$.

STEP 2. Substitute $a(t)$, $b(t)$, and $r(t)$ into Eq. 3.2 to get the equation of the envelope for the family of circles with radius $r(t)$ and with the center $(a(t),b(t))$.

STEP 3. If the equation for the envelope from STEP 2 is "close" enough to the boundary of the given object, then stop. We have found approximate medial axis $\{p_0(a_0,b_0),p_1(a_1,b_1),\cdots,p_n(a_n,b_n)\}$.

Otherwise, find another medial axis point and repeat from STEP 1.

STEP 1 requires curve fitting of the data. There are various curve fitting techniques one can choose from. We chose natural cubic splines with zero curvature at each end point.

One problem in this case is that the values of t for each data points are not known. We used a common technique which computes the succesive values of t_i from the distance between two succesive points, $p_{i-1}(a_{i-1},b_{i-1})$ and $p_i(a_i,b_i)$ (see [10]). Now the parameters of natural cubic splines functions can be easily computed by solving a linear simultaneous equation involving a matrix of tri-diagonal form by Gaussian elimination.

STEP 2 was explained in details in Section 3. In STEP 3 we need a definition of the distance of two functions to determine if two functions are close to each other. one can use the standard L^2 norms that define the distance of two functions as follows:

$$d(f,g) = \int_{t_1}^{t_2} (f(t) - g(t))^2 dt$$

Now Figure 4.2 shows the implementation results of our methods. As it is shown, our methods produces the good result.

5. Conclusions

We proposed a new approach for constructing the medial axis of a 2D object with smooth boundary. Our approach constructs a finite number of inscribed circles, each time starting from an inside point. We keep adding another maximal disk until the envelope of the set of inscribed circles is close enough to the original boundary of the object. Previous works constructed the medial axis from the boundary points either by thinning algorithms or by emnating wave front perpendicularly from the boundary points. Therefore their result was highly sensitive to a small irregularities of the object's boundary curve. However, our approach constructs medial axis from an inside point. So our approach produces robust results that ignore small perturbations of the boundary curve and highlight only the essential strutural features. We will investigate the stability properties of our method and the medial axis transformation for objects with more general shape in the future.

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