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The Realization of Binary Systems by the Latent Stress Model

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Abstract

We study under what condition the probability of the failure state of each component is realized by the latent stress model which considers the cause of the failure as well as the state of the given system. As a result, when the number of the components is less than or equal to 3, the MTP_2 property is a necessary and sufficient condition for the realization of the probability of system state by the latent stress model. Moreover, if the probability of the system state involving 2 or 3 components satisfies the MTP_2 property, one could guess that each component is under the same stress.

1. Introduction

Suppose there is a system involving n components. To a single component, we associate a random variable X. It has a value 1 if a component is failing and 0 if a component is functioning.

For n-component system, the state of the system can be described by an n-dimensional random vector $\boldsymbol{X}=(X_1, \cdots, X_n)$, where $X_j=1$ or 0 depending on whether the j-th component is failing or not. We call $\boldsymbol{X}$ a *system state vector*.

Let Θ describe the stress of the component such as heat, fraction, or even unobserved resistance. As well as the state of the given system, one would be interested in the cause of failure. Thus, the latent stress model is valuable in analyzing the system fault.

The latent stress model represents the distribution of the system state vector $\boldsymbol{X}$ as

$$P(\boldsymbol{X}=\boldsymbol{x})=\int_{-\infty}^{\infty} \prod_{j=1}^{n} p_j(\theta)^{x_j} (1-p_j(\theta))^{1-x_j} \, dF(\theta), \tag{1}$$

where $dF(\theta)$ is a probability measure of stress variable Θ and each functions $p_j(\theta)$ is nondecreasing in θ.

The model (1) is represented under the following three assumptions:

ⅰ. latent conditional independence

$$P(\boldsymbol{X}=\boldsymbol{x} \mid \theta)=\prod_{j=1}^{n} P(X_j=x_j \mid \theta).$$

That is, the latent stress variable is so informative that the distribution of the system state vector is conditionally independent, given θ.

ⅱ. latent monotonicity

$$p_j(\theta)=P(X_j=1 \mid \theta) \text{ is nondecreasing in } \theta.$$

In other words, we assume that a component with higher stress is more likely to be failed well than one with lower stress.

ⅲ. latent unidimensionality

The latent stress variable is unidimensional.

The latent stress model (1) is originally from item response theory studied by many psychometricians, Cressie and Holland(1983), Holland and Rosenbaum (1986), and Junker(1993). They call (1) *the latent trait model* expressing the distribution of item response vector $\boldsymbol{X}$ where either $X_j=1$ if an examinee answers j-th item correctly, or $X_j=0$ otherwise. Also they denote that Θ is a latent trait variable which is examinee's ability.

As is well known, under the three conditions, each component of a system is not independent. Esary and Proschan(1970) studied the binary system which is not independent. The latent stress model with equal failing probability of each component was researched by Lau(1992).

In Section 2, we review several known probability inequalities of $P(\boldsymbol{X}=\boldsymbol{x})$ realized by the latent stress model. Also we introduce a notion $CMTP_2$ which is stronger than the MTP_2 property.

In Section 3, we discuss the $CMTP_2$ property, essentially a conditional MTP property.

In Section 4 (see Theorem 4.1 and Theorem 4.2), we prove a necessary and sufficient condition for the realizability in terms of $P(\boldsymbol{X}=\boldsymbol{x})$ of a latent stress model, when a system has 3 components.

Since the MTP_2 property will be shown to be a necessary and sufficient condition for the realizability of $P(\boldsymbol{X}=\boldsymbol{x})$ by the latent stress model when $n=3$, a stronger necessary condition such as $CMTP_2$ must follow from the MTP_2 property in the case that $n=3$.

2. Notions of positive ordering

Let

$$(\Omega_j, A_j, P_j) \text{ for } j=1, \cdots, n$$

be given probability measure spaces and let

$$(\Omega, A, P)=\prod_{j=1}^{n}(\Omega_j, A_j, P_j)$$

be their direct product. Further, let $\boldsymbol{X}=(X_1, \cdots, X_n)$ be a random vector with $X_j \in \Omega_j (j=1, \cdots, n)$. Also let $\lambda=\prod_{j=1}^{n}\lambda_j$ denote a product measure on Ω, where λ_j is σ-finite measure on Ω_j, $j=1, \cdots, n$.

2.1 Total positivity

As a strong form of positive ordering, the notion of total positivity plays an important role in the theory of convexity, inequalities, and moment problems.

Definition 2.1 *A real function ϕ is totally positive of order $r(TP_r)$ if*

$$\phi\begin{pmatrix} x_1, x\ , \cdots, x_m \\ y_1, y_2, \cdots, y_m \end{pmatrix}=\begin{vmatrix} \phi(x_1, y_1) & \phi(x_1, y_2) & \cdots & \phi(x_1, y_m) \\ \phi(x_2, y_1) & \phi(x_2, y_2) & \cdots & \phi(x_1, y_m) \\ \vdots & \vdots & \vdots & \vdots \\ \phi(x_m, y_1) & \phi(x_m, y_2) & \cdots & \phi(x_m, y_m) \end{vmatrix} \geq 0$$

for all

$$x_1<x_2<\cdots<x_m; y_1<y_2<\cdots<y_m \text{ and } 1 \leq m \leq r.$$

2.2 Multivariate total positivity of order 2

For $\boldsymbol{X}=(X_1, \cdots, X_n)$, $\boldsymbol{Y}=(Y_1, \cdots, Y_n)$ defined on Ω, we define the following operations

$$\boldsymbol{x} \vee \boldsymbol{y} = (\max(x_1, y_1), \cdots, \max(x_n, y_n))$$

and

$$\boldsymbol{x} \wedge \boldsymbol{y} = (\min(x_1, y_1), \cdots, \min(x_n, y_n))$$

Multivariate version of total positivity of order 2 can be defined as follows.

Definition 2.2 *A random vector $X=(X_1, \cdots, X_n)$ defined on Ω having the density function $f=dP/d\lambda$ is multivariate totally positive of order 2(MTP_2) if*

$$f(\boldsymbol{x} \vee \boldsymbol{y}) f(\boldsymbol{x} \wedge \boldsymbol{y}) \geq f(\boldsymbol{x}) f(\boldsymbol{y}). \tag{2}$$

In general, if a function f satisfies (2), we say that f has the MTP_2 property.

Many fundamental properties of MTP_2 were developed by Karlin and Rinott (1980).

Proposition 2.1 *Let $(\boldsymbol{y}_1, \boldsymbol{z})$ and $(\boldsymbol{z}, \boldsymbol{y}_2)$ be partitions of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, respectively. Suppose $f(\boldsymbol{x}_1)=f(\boldsymbol{y}_1, \boldsymbol{z})$ and $g(\boldsymbol{x}_2)=g(\boldsymbol{z}, \boldsymbol{y}_2)$ satisfy the MTP_2 condition. Then, also*

$$h(\boldsymbol{y}_1, \boldsymbol{y}_2) = \int f(\boldsymbol{y}_1, \boldsymbol{z}) g(\boldsymbol{z}, \boldsymbol{y}_2) d\boldsymbol{z}$$

satisfies the MTP_2 condition.

Proposition 2.2 *If $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ have the MTP_2 property, then $f(\boldsymbol{x})g(\boldsymbol{x})$ also has the MTP_2 property.*

Kemperman(1977) found the relationship between TP_2 and MTP_2.

Proposition 2.3 *Let $f(\boldsymbol{x})$ be TP_2 in every pair of arguments. Suppose $f(\boldsymbol{x})>0$ for all $\boldsymbol{x} \in \Omega$. Then, the function $f(\boldsymbol{x})$ has the MTP_2 property for all $\boldsymbol{x} \in \Omega$.*

2.3 Stronger concept of positive dependence orderings, $CMTP_2$

Holland and Rosenbaum(1986) defined *conditional multivariate totally positivity of order 2* ($CMTP_2$) property which is stronger than the MTP_2 property.

Let $(\boldsymbol{Y}, \boldsymbol{Z})$ be a partition of $\boldsymbol{X}$. More precisely,

$$\boldsymbol{Y} = \{X_j : j \in I\} \text{ and } \boldsymbol{Z} = \{X_j : j \in J\}, \tag{3}$$

where I and J are complementary subsets of $N = \{1, \cdots, n\}$.
The definition of $CMTP_2$ is as follows.

Definition 2.4 *A random vector* $\boldsymbol{X}$ *has the* $CMTP_2$ *property if a conditional density* $\boldsymbol{X}$ *given* $\boldsymbol{Z}$ *has* MTP_2 *property for all possible partition* $(\boldsymbol{Y}, \boldsymbol{Z})$ *of* $\boldsymbol{X}$ *and any* $Z \in \sum_{j \in J} A_j$.

2.4 Properties of a latent stress model

In Section 1, we introduced three basic assumptions of a latent stress model. The assumptions induce many interesting necessary conditions.

By the assumption of unidimensionality and monotonicity of the latent stress variable, $p_j(\theta)/(1-p_j(\theta))$ is nondecreasing in θ, that is,

$$\frac{P(X_j=1 \mid \theta')P(X_j=0 \mid \theta)}{P(X_j=1 \mid \theta)P(X_j=0 \mid \theta')} \geq 1 \tag{4}$$

for all $\theta < \theta'$. But, (4) is equivalent to

$$\begin{vmatrix} P(X_j=0 \mid \theta) & P(X_j=0 \mid \theta') \\ P(X_j=1 \mid \theta) & P(X_j=1 \mid \theta') \end{vmatrix} \geq 0.$$

Thus, $P(X_j=x_j \mid \Theta=\theta)$ has the TP_2 property. Hence, from Proposition 2.2, it is immediate that $\prod_{j=1}^{n} P(X_j=x_j \mid \theta)$ has the MTP_2 property. Therefore, by Proposition 2.1,

$$P(\boldsymbol{X}=\boldsymbol{x}) = \int_{-\infty}^{\infty} \prod_{j=1}^{n} P(X_j=x_j \mid \theta)dF(\theta)$$

satisfies the MTP_2 property.

X_1	X_2	$\Theta=\theta$	$\Theta=\theta'$
0	0	0.	0.1
0	1	0.2	0.1
1	0	0.1	0.1
1	1	0.	0.4

⟨ Table 1 ⟩ joint density of (x_1, x_2, θ)

Remar 2.5 Let (X_1, X_2, θ) be a random vector such that each density function of (X_1, θ) and (X_2, θ) satisfies the TP_2 property. Then, (X_1, X_2) need not have the TP_2 property. See the counterexample described in Table 1.

Rosenbaum(1984) showed that $P(\boldsymbol{X}=\boldsymbol{x})$ represented by a latent stress model always has the $CMTP_2$ property, see also Holland and Rosenbaum(1986).

3. General properties of the latent stress model

Let $\boldsymbol{X}=(X_1, \cdots, X_n)$ be a system state vector. Here $X_j=1$ or 0 depending on whether a j-th component is failing or not. We would like to know what kind of $P(\boldsymbol{X}=\boldsymbol{x})$ is realizable by a latent stress model. Cressie and Holland (1983) studied necessary and sufficient conditions for the realizability of the latent stress (trait) model when there are two or three components ($n\leq 3$).

To avoid trivial details, we will usually assume that $0<p_j=P(X_j=1 \mid \theta)<1$. This implies that $P(\boldsymbol{X}=\boldsymbol{x})>0$ for all $\boldsymbol{x}\in\{0, 1\}^n$.

For convenience, we let

$$V_j(\theta) = \frac{p_j(\theta)}{1-p_j(\theta)}.$$

Here $p_j(\theta)=P(X_j=1 \mid \theta)$. Note that $0<V_j(\theta)<\infty$. Thus, the distribution of the system state vector $\boldsymbol{X}$ becomes

$$P(\boldsymbol{X}=\boldsymbol{x})=\int_{-\infty}^{\infty} h(\boldsymbol{x}, \theta)dF(\theta). \tag{5}$$

Here,

$$h(\boldsymbol{x}, \theta) = C(\theta)\prod_{j=1}^{n} V_j(\theta)^{x_j},$$

where

$$C(\theta) = \prod_{j=1}^{n} P(X_j = 0 \mid \theta) = \sum_{j=1}^{n} (1 - p_j(\theta)).$$

Observe that each $V_j(\theta)$ is nondecreasing in θ.

Let A be a subset of $N = \{1, \cdots, n\}$. Let further, $q(A)$ denote the probability that the set of failed components coincides with A. Thus,

$$q(A) = P(\boldsymbol{X} = \boldsymbol{x}), \textit{ where } x_j = 1 \text{ if and } \textit{only} \text{ if } j \in A. \tag{6}$$

Formula (5) implies that

$$q(A) = \int_{-\infty}^{\infty} \prod_{j \in A} V_j(\theta) dG(\theta), \tag{7}$$

where $dG(\theta) = C(\theta)dF(\theta)$. Note that $\sum_{A \subset N} q(A) = 1$ and $q(\phi) = \int C(\theta)dF(\theta)$.

Moreover, $q(A) > 0$ for all $A \subset N$.

The following result is due to Holland and Rosenbaum(1986). It states that under the latent stress model the function $q(A)$ has the MTP_2 property. We include a new proof

Proposition 3.1 *Let A and B be arbitrary subsets of $N = \{1, \cdots, n\}$. Then,*

$$q(A \cup B)q(A \cap B) \geq q(A)q(B). \tag{8}$$

Proof. For any pair of nondecreasing functions, f and g on $\mathbf{R}$ and any finite measure μ on $\mathbf{R}$, one has that

$$\int\int (f(x) - f(x'))(g(x) - g(x'))\, d\mu(x)\, d\mu(x') \geq 0,$$

and thus,

$$\int fg d\mu \int d\mu \geq \int f d\mu \int g d\mu.$$

Letting,

$$f(\theta) = \prod_{j\in A\setminus B} V_j(\theta)\,;\; g(\theta) = \prod_{j\in B\setminus A} V_j(\theta)\,;\; d\mu(\theta) = \prod_{j\in A\cap B} V_j(\theta)dG(\theta),$$

(8) is immediate. □

The representation (6) and Proposition 3.1 imply that $P(\boldsymbol{X}=\boldsymbol{x})$ has the MTP_2 property as was shown by Holland and Rosenbaum(1986).

Let $(\boldsymbol{Y}, \boldsymbol{Z})$ be a partition of $\boldsymbol{X}$, defined in (3). Let, further $Z\subset\{0, 1\}^J$. Define

$$\begin{aligned}P(\boldsymbol{y}\mid Z) &= P(\boldsymbol{Y}=\boldsymbol{y}\mid \boldsymbol{z}\in \boldsymbol{Z})\\ &= \frac{1}{P(\boldsymbol{z}\in \boldsymbol{Z})}\sum_{z\in Z} P(\boldsymbol{Y}=\boldsymbol{y},\, \boldsymbol{Z}=\boldsymbol{z}).\end{aligned}$$

In Section 2, we reviewed the $CMTP_2$ property. It states that

$$P(\boldsymbol{y}\vee\boldsymbol{y}^*\mid Z)P(\boldsymbol{y}\wedge\boldsymbol{y}^*\mid \boldsymbol{Z})\geq P(\boldsymbol{y}\mid Z)P(\boldsymbol{y}^*\mid \boldsymbol{Z})$$

for all $\boldsymbol{y}, \boldsymbol{y}^*\in\{0, 1\}^I$ and all $Z\subset\{0, 1\}^J$.

In view of Proposition 3.1, the $CMTP_2$ property follows immediately from the following lemma.

Lemma 3.1 *Let* $1\leq m\leq n-1$. *If each latent stress model with* m *components satisfies a certain property, the same property holds for each conditional probability of the form* $P(\boldsymbol{Y}=\boldsymbol{y}\mid \boldsymbol{z}\in \boldsymbol{Z})$ *with* $|I|=m$, *associated to a latent stress model involving* n *components.*

Proof. The statement follows immediately from the following representation:

$$\begin{aligned}&P(\boldsymbol{Y}=\boldsymbol{y}\mid \boldsymbol{z}\in \boldsymbol{Z})\\ &= \frac{1}{P(\boldsymbol{z}\in \boldsymbol{Z})}\int_{-\infty}^{\infty}\prod_{j\in I} p_j(\theta)^{y_j}(1-p_j(\theta))^{1-y_j}\sum_{z\in Z}\prod_{l\in J} p_l(\theta)^{z_l}(1-p_l(\theta))^{1-z_l}dF(\theta)\\ &= \int_{-\infty}^{\infty}\prod_{j\in I} p_j(\theta)^{y_j}(1-p_j(\theta))^{1-y_j}dH(\theta)\end{aligned}$$

for all $\boldsymbol{y}\in\{0, 1\}^I$. Here, the distribution function $dH(\theta)$ is defined by

$$dH(\theta) = \frac{1}{P(z\in Z)}\sum_{z\in Z}\prod_{j\in J} p_j(\theta)^{z_j}(1-p_j(\theta))^{1-z_j}dF(\theta). \tag{9}$$

□

4. Conditions for realizability

In this section, we assume that $n \geq 3$.

The MTP_2 property of $q(A)$ (defined as (7)) requires that

$$q(A \cup B) q(A \cap B) \geq q(A) q(B)$$

for an arbitrary A, $B \subset N$. For convenience, we define

$$\tilde{q}(A) = \frac{q(A)}{q(\phi)}, \; (A \subset N). \tag{10}$$

Thus, $\tilde{q}(\phi) = 1$. If $n = 2$, the MTP_2 property is equivalent to

$$\tilde{q}(\{1, 2\}) \geq \tilde{q}(\{1\}) q(\{2\}).$$

When $n = 3$, the MTP_2 property is equivalent to:

$$\begin{aligned} &\tilde{q}(\{i, j\}) \geq \tilde{q}(\{i\}) \tilde{q}(\{j\}) \text{ if } i \neq j \text{ and } \{i, j\} \subset N; \\ &\tilde{q}(\{1, 2, 3\}) q(\{k\}) \geq \tilde{q}(\{i, k\}) \text{ if } i, j, k \textit{ are distinct}. \end{aligned} \tag{11}$$

Theorem 4.1 *Let $n = 2$. In order that $q(A)$, $(A \subset N)$ can be realized by a latent stress model, it is necessary and sufficient that*

$$\tilde{q}(\{1, 2\}) \geq \tilde{q}(\{1\}) \tilde{q}(\{2\}). \tag{12}$$

Furthermore, if (12) holds, $q(A)$, $(A \subset N)$ can be realized by a probability measure with 2-point support.

Proof. We need that

$$q(\{j\}) = \int V_j(\theta) dG'(\theta), \; (j = 1, 2) \; ; \tilde{q}(\{1, 2\}) = \int V_1(\theta) V_2(\theta) dG'(\theta),$$

where $dG'(\theta) = dG(\theta) / q(\phi)$ is a suitable probability measure and where each function $V_j(\theta)$, $(j = 1, 2)$ is nondecreasing in θ. Since

$$\int \int (V_1(\theta_1) - V_1(\theta_2))(V_2(\theta_1) - V_2(\theta_2)) dG'(\theta_1) dG'(\theta_2) \geq 0,$$

we see that $2(\tilde{q}(\{1,2\}) - \tilde{q}(\{1\})q(\{2\}) \geq 0$, which is (12).

Conversely, assume (12), where $\tilde{q}(\{i\})>0$, $(i=1, 2)$. Let $\theta_1<\theta_2$, $0<p\leq 1$; (the parameter p will be specified below). Further, let be u and v such that $0\leq u<\tilde{q}(\{1\})$ and $0\leq v<\tilde{q}(\{2\})$, and define

$$V_1(\theta_1) = u\ ;\ V_2(\theta_1)=v\ ;$$

$$V_1(\theta_2) = u + \frac{q(\{1\})-u}{p}\ ;\ V_2(\theta_2 = v + \frac{q(\{2\})-v}{p}$$

Let further $G'(\theta)$ have support $\{\theta_1, \theta_2\}$ with masses $1-p$ and p, respectively. One easily verifies that

$$(1-p)V_j(\theta_1) + pV_j(\theta_2) = \tilde{q}(\{j\}),\ (j=1, 2) \tag{13}$$

We further need that

$$\begin{aligned}\tilde{q}(\{1,2\}) &= (1-p)V_1(\theta_1)V_2(\theta_1) + pV_1(\theta_2)V_2(\theta_2)\\ &= \frac{(q(\{1\})-u)(q(\{2\})-v)}{p}\\ &\quad + uv + u(\tilde{q}(\{2\})-v) + v(\tilde{q}(\{1\})-u).\end{aligned} \tag{14}$$

This is equivalent to

$$p= \frac{(q(\{1\})-u)(q(\{2\})-v)}{(q(\{1,2\})-q(\{1\})q(\{2\})) + q(\{1\})-u)(q(\{2\})-v)}\ . \tag{15}$$

Note that $\tilde{q}(\{1,2\}) \geq \tilde{q}(\{1\})\tilde{q}(\{2\})$, $u<\tilde{q}(\{1\})$ and $v<\tilde{q}(\{2\})$ implies $0<p\leq 1$.

Let us now study the case $n=3$.

Theorem 4.2 *Let $n=3$. The MTP_2 condition (as in (11)) is not only necessary but also sufficient for the realization of $P(\boldsymbol{X}=\boldsymbol{x})$ by a latent stress model. Moreover, if it holds then $P(\boldsymbol{X}=\boldsymbol{x})$ can even be realized by a probability measure with 2-point support.*

Proof. In Proposition 3.1, we already showed that $P(\boldsymbol{X}=\boldsymbol{x})$ represented by the latent stress model has the MTP_2 property. So the proof for the sufficiency of the MTP_2 property is only left.

Let

$$b_{ij} = \frac{q(\{i, j\})}{q(\{i\})q(\{j\})}, \quad (1 \le i < j \le 3) ; \quad c = \frac{q(\{1, 2, 3\})}{q(\{1\})q(\{2\})q(\{3\})}.$$

Then, the MTP_2 condition is equivalent to

$$1 \le b_{ij} \le c, \quad (1 \le i < j \le 3) ; \tag{16}$$

$$b_{ik} b_{jk} \le c, \quad (i,\ j,\ k \textit{ are distinct with } i < j). \tag{17}$$

Permutating indices, we may assume without loss of generality that

$$1 \le b_{12} \le b_{13} \le b_{23}. \tag{18}$$

In this case, the inequality $c \ge b_{13} b_{23}$ implies analogous inequalities such as $c \ge b_{12} b_{23}$ and $c \ge b_{12} b_{13}$.

Let $\theta_1 < \theta_2$ and $0 < p < 1$. Further, let dG' have mass p and $q = 1 - p$ at θ_1 and θ_2 respectively. Letting $U_{ij} = V_i(\theta_j)/\tilde{q}(\{i\})$, *we need that*

$$pU_{i1} + qU_{i2} = 1, \quad (i = 1, 2, 3) \tag{19}$$

$$pU_{i1}U_{j1} + qU_{i2}U_{j2} = b_{ij}, \quad (1 \le i < j \le 3) \tag{20}$$

$$pU_{11}U_{21}U_{31} + qU_{12}U_{22}U_{32} = c \tag{21}$$

and further that $0 \le U_{i1} \le U_{i2}$. Thus (19) implies that

$$p(1 - U_{i1}) = q(U_{i2} - 1) = y_i \textit{(say)}, \ (i = 1, 2, 3), \textit{ where } 0 \le U_{i1} \le 1 \le U_{i2}.$$

Therefore,

$$U_{i1} = 1 - \frac{y_i}{p} \text{ and } U_{i2} = 1 + \frac{y_i}{q} \textit{ where } 0 \le y_i \le p, \quad (i = 1, 2, 3). \tag{22}$$

Let further

$$\beta_{ij} = b_{ij} - 1, \quad (1 \le i < j \le 3).$$

Thus, in view of (18), the β_{ij} are prescribed numbers satisfying

$$0 \leq \beta_{12} \leq \beta_{13} \leq \beta_{23}. \tag{23}$$

Moreover, the condition $c \geq b_{13} b_{23}$ takes the form

$$c \geq 1 + \beta_{13} + \beta_{23} + \beta_{13} + \beta_{13}\beta_{23} \tag{24}$$

For $1 \leq i < j \leq 3$, (20) is equivalent to

$$\begin{aligned}\beta_{ij} &= b_{ij} - 1 = pU_{i1}U_{j1} + qU_{i1}U_{j2} - 1 \\ &= p\left(1 - \frac{y_i}{p}\right)\left(1 - \frac{y_j}{p}\right) + q\left(1 + \frac{y_i}{q}\right)\left(1 + \frac{y_j}{q}\right) - 1 \\ &= \frac{y_i y_j}{pq}.\end{aligned}$$

Thus, we need that

$$y_1 = \sqrt{pq\beta_{12}\beta_{13}\beta_{23}}\ ;\ y_2 = \sqrt{pq\beta_{12}\beta_{23}/\beta_{13}}\ ;\ y_3 = \sqrt{pq\beta_{13}\beta_{23}\beta_{12}}\ . \tag{25}$$

In view of (23), one has that $y_1 \leq y_2 \leq y_3$. Thus, the condition $y_i \leq p$ for $i = 1, 2, 3$ is implied by $y_3 \leq p$, that is,

$$\frac{p}{q} \geq \frac{\beta_{13}\beta_{23}}{\beta_{12}}. \tag{26}$$

Using (22), condition (21) takes the form

$$\begin{aligned}c &= p\left(1 - \frac{y_1}{p}\right)\left(1 - \frac{y_2}{p}\right)\left(1 - \frac{y_3}{p}\right) + q\left(1 + \frac{y_1}{q}\right)\left(1 + \frac{y_2}{q}\right)\left(1 + \frac{y_3}{q}\right) \\ &= 1 + (y_1 y_2 + y_1 y_3 + y_2 y_3)\left(\frac{1}{p} + \frac{1}{q}\right) + y_1 y_2 y_3\left(-\frac{1}{p^2} + \frac{1}{q^2}\right).\end{aligned}$$

Using (23), this becomes

$$1 + (\beta_{12} + \beta_{13} + \beta_{23}) + \left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)\sqrt{\beta_{12}\beta_{13}\beta_{23}} = c. \tag{27}$$

Here, the left-hand side is a continuous increasing function of p/q varying from the value $1+\beta_{13}+\beta_{23}+\beta_{13}\beta_{23}$ (when $p/q=\beta_{13}\beta_{23}/\beta_{12}$) to the value $+\infty$ (when $q=0$). Hence, (27) has a unique solution satisfying (26). □

As we have seen in Theorem 4.2, the MTP_2 condition is sufficient when $n=3$. Thus, in this special case, the stronger necessary condition $CMTP_2$ as in Lemma 3.1 is implied by the MTP_2 condition.

Corollary 4.3 When $n=3$, the two conditions, MTP_2 and $CMTP_2$ are equivalent.

5. Conclusion and remarks

As we have seen, the latent stress model forms the mixture of the conditional distribution of the system state vector, given the latent stress value. That is, the latent stress model considers the cause of failure of each component.

The three assumptions for the model induce known inequalities such as the MTP_2 property and the $CMTP_2$ property.

When the system contains 3 components, the MTP_2 property is a necessary and sufficient condition for the realization of $P(\boldsymbol{X}=\boldsymbol{x})$ by the latent stress model. More precisely, if the distribution of the system state vector, $P(\boldsymbol{X}=\boldsymbol{x})$ satisfies the MTP_2 property, one would suspect that each component be failed by the same cause such as heat or resistance.

When $n\geq 4$, the MTP_2 is not sufficient. The counter example is founded by Holland and Rosenbaum(1986). Also Choi(1994) shows that the $CMTP_2$ property which is a stronger notion than the MTP_2 property is not sufficient for the realization of $P(\boldsymbol{X}=\boldsymbol{x})$ by the latent stress model.

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