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FUZZY SEMI–INNER–PRODUCT SPACE

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ABSTRACT. G.Lumer [8] introduced the concept of semi-product space. H.M.El-Hamouly [7] introduced the concept of fuzzy inner product spaces.

In this paper, we defined fuzzy semi-inner-prodct space and investigated some properties of fuzzy semi product space.

1. Preliminaries

Definition 1.1 [1]. *A fuzzy real number ζ is a nonascending, left continuous function from R into $I = [0,1]$ with $\zeta(-\infty^+) = 1$ and $\zeta(+\infty^-) = 0$. The set of all fuzzy real numbers will be denoted by $R(I)$. The partial ordering $\geq$ on $R(I)$ is the natural ordering of real functions. The set of all reals R is canonically embedded in $R(I)$ in the following fashion, for every $r \in R$, we associated the fuzzy real number $\bar{r} \in R(I)$ which is defined by*

$$\bar{r} = \begin{cases} 1 & if\, t \leq r \\ 0 & if\, t > r \end{cases}$$

The set $R^*(I)$ of all *nonnegative* real numbers is defined by

$$R^*(I) = \{\zeta \in R(I) : \zeta \geq \bar{0}\}.$$

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Definition 1.2 [5]. *Let ζ, ξ be two fuzzy real numbers in $R(I)$, and let s be any real number. Then*

(i) Addition of fuzzy real numbers $\oplus$ is defined on $R(I)$ by

$$(\xi \oplus \zeta)(s) = \sup\{\xi(t) \wedge \zeta(s-t) : t \in R\}.$$

(ii) Scalar multiplication by a nonnegative $r \in R$ is defined on $R(I)$ by

$$(r\xi)(s) = \begin{cases} \bar{0} & \text{if } r = 0 \\ \xi(\frac{s}{r}) & \text{if } r > 0 \end{cases}$$

It is well known that the above two operations are well defined on $R(I)$ and the canonical embedding of R in $R(I)$ preserves these two operations. The results in the following proposition are well known.

Proposition 1.3 [4].

(i) Addition and scalar multiplication preserve the order $\geq$ on $R(I)$.

(ii) $R(I)$ is closed under these two operations.

(iii) For η, ζ and $\xi \in R(I)$, we have

$$\eta \oplus \xi \geq \zeta \oplus \xi \text{ iff } \eta \geq \zeta.$$

Definition 1.4 [6]. *Multiplication of two nonnegative fuzzy real numbers η , $\zeta \in R^*(I)$ is defined by*

$$(\eta\zeta)(s) = \begin{cases} 1 & \text{if } s \leq 0 \\ \sup\{\eta(b) \wedge \zeta(\frac{s}{b}) : b > 0\} & \text{if } s > 0, \end{cases}$$

where $s \in R$. it is shown in [6] that $R^*(I)$ is closed under multiplication.

Definition 1.5 [2]. *A fuzzy pseudo-norm on a real of complex space X is a function $\| \; \| : X \to R^*(I)$ which satisfies the following two conditions ; for $x, y \in X$ and s in the field*

(i) $\|sx\| = |s|\|x\|$,

(ii) $\|x+y\| \leq \|x\| \oplus \|y\|$.

The algebraic properties of addition and nonnegative scalar multiplication on $R^*(I)$ enable us to embed $R^*(I)$ in the smallest real vector space $M(I)$ as follow.

Definition 1.6 [3]. *The set $M(I)$ is the cartesian product $R^*(I) \times R^*(I)$ modulo the equivalence relation $\sim$ defined by*

$$(\eta, \zeta) \sim (\xi, \lambda) \text{ iff } \eta \oplus \lambda = \zeta \oplus \xi.$$

The partial order $\geq$ on $M(I)$ is defined by

$$(\eta, \zeta) \geq (\xi, \lambda) \text{ iff } \eta \oplus \lambda \geq \zeta \oplus \xi.$$

The set $M^*(I)$ is defined by

$$\begin{aligned} M^*(I) &= \{(\eta, \zeta) \in M(I) : (\eta, \zeta) \geq \bar{0}\} \\ &= \{(\eta, \zeta) \in M(I) : \eta \geq \zeta\} \end{aligned}$$

$R^*(I)$ is canonically embedded in $M(I)$ by representing each $\eta \in R^*(I)$ as $(\eta, \bar{0}) \in M(I)$. Also R is embedded in $M(I)$ as follows : for $r \in R$, r is identified with $(\bar{r}, \bar{0}) \in M(I)$ if $r \geq 0$ and with $(\bar{0}, (\overline{-r})) \in M(I)$ if $r < 0$.

Addition $\oplus$ and real scalar multiplication are defined on $M(I)$ by :

$$(\eta, \zeta) \oplus (\xi, \lambda) = (\eta \oplus \xi, \zeta \oplus \lambda), \tag{i}$$

$$t(\eta, \zeta) \begin{cases} (t\eta, t\zeta) & \text{if } t \geq 0 \\ (|t|\zeta, |t|\eta) & \text{if } t < 0. \end{cases} \tag{ii}$$

THeorem 1.7 [3]. *The above addition and scalar multiplication are well defined on $M(I)$. Under these twooperations, $M(I)$ is the smallest real vector pace including $R^*(I)$. In patricular, the canonical embeding of $R^*(I)$ into $M(I)$ preserves addition and nonnegative scalar multiplication, while the canonical embedding of R into $M(I)$ is a vector space embedding.*

Definition 1.8 [3]. *The N-Euclidean norm on $M(I)$ is the fuzzy pseudo-norm $||\lfloor \; \rfloor||$ defined by : for $(\eta, \zeta) \in M(I)$,*

$$\begin{aligned} ||\lfloor(\eta,\zeta)\rfloor|| &= inf\{\xi \in R^*(I) : \xi \geq (\zeta,\eta) and\ \xi \geq (zeta,\eta)\} \\ &= inf\{\xi \in R^*(I) : \xi \oplus \zeta \geq \eta \text{ and } \xi \oplus \eta \geq \zeta\}, \end{aligned}$$

where $\xi = (\xi, \bar{0})$ according to the embeding of $R^*(I)$ in $M(I)$.

Definition 1.9 [4]. *A real algebra X with a fuzzy pseudo-norm $\|\ \|$ on X will be called a fuzzy pseudo-norm algebra if for all $x, y \in X$,*

$$\|xy\| \leq \|x\|\|y\|,$$

where multiplication in the right-hand side is the fuzzy multiplication on $R^*(I)$.

Definition 1.10 [4]. *Multiplication on $M(I)$ is defined by:*
for (η,ζ), $(\xi,\lambda) \in M(I)$,

$$(\eta,\zeta)(\xi,\lambda) = (\eta\xi \oplus \zeta\lambda, \eta\lambda \oplus \zeta\xi).$$

Theorem 1.11 [4].

(i) *Multiplication on $M(I)$ is well defined.*
(ii) *The canonical embeddings of $R^*(I)$ and R into $M(I)$ preserve multiplication.*
(iii) *Underaddition, scalar multiplication, $M(I)$ is a real associative and commutative algebra with unit element $\bar{1} = (\bar{1}, \bar{0})$.*

(iv) *$M(I)$ is not an integral domain.*

(v) *$(M(I), \|\lfloor\ \rfloor\|)$ is a fuzzy pseudo-normed vector space and is a fuzzy pseudo-normed algebra under its multiplication.*

Definition 1.12. *Let U be a fuzzy subset of a universe X and let $\alpha \in I_1 = [0, 1)$. The α -cut of U is the crisp subset of X*

$$U^{(\alpha)} = \{x \in X : U(x) > \alpha\}.$$

Fuzzy real numbers in $R^*(I)$ can be considered as fuzzy subsets of the set R^* of all nonnegative reals. Therefore, for each $\eta \in R(I)$, its α -cut $\eta^{(\alpha)} = [0, t)$ or $= [0, t]$, where $t = \vee\{x \in R : \eta(x) > \alpha\}$ is uniquely identified with the number t. It is obvious that α-cuts preserve the three operations on $R^*(I)$ and order on $R^*(I)$ in the following sense : for every η, $\zeta \in R^*(I), \alpha \in I_1$, and $r \geq 0$ we have

$$(\eta \oplus \zeta)^{(\alpha)} = \eta^{(\alpha)} + \zeta^{(\alpha)} \tag{i}$$

$$(r\eta)^{(\alpha)} = r\eta^{(\alpha)} \tag{ii}$$

$$(\eta\zeta)^{(\alpha)} = \eta^{(\alpha)}\zeta^{(\alpha)} \tag{iii}$$

$$\eta \leq \zeta \text{ iff } \eta^{(\alpha)} \leq \zeta^{(\alpha)}, \forall \alpha \in I_1. \tag{iv}$$

Proposition 1.13 [4].

(i) *For $(\eta, \zeta) \in R^*(I)$, $\eta^2 < \zeta^2$ iff $\eta < \zeta$.*

(ii) *For every $\eta \in R^*(I)$, there exists a unique square root ξ in $R^*(I)$ such that $\xi^2 = \eta$.*

(iii) *For $(\eta, \zeta) \in M(I)$ we have $(\eta, \zeta)^2 \in M^*(I)$.*

(iv) For $(\eta, \zeta) \in M(I)$ *we have* $|\lfloor(\eta,\zeta)^2\rfloor| = |\lfloor(\eta,\zeta)\rfloor|^2$.

2. Fuzzy Semi-Inner-Product Space

In this section, we will define the fuzzy semi-inner-product and establish some properties that goes with it. First we introduce the notation of the α-cuts of the fuzzy real numbers to $M(I)$.

Definition 2.1. *Let* $(\eta,\zeta) \in M(I)$ *and* $\alpha \in I_1$. *We define the* α*-cut of* (η,ζ) *to be the real number*

$$(\eta,\zeta)^{(\alpha)} = \eta^{(\alpha)} - \zeta^{(\alpha)}.$$

Proposition 2.2 [7].

(i) The α*-cut* $(\eta,\zeta)^{(\alpha)}$ *is well-defined on* $M(I)$.

(ii) $(\eta,\zeta) = (\xi,\lambda)$ *in* $M(I)$ *iff they have the same indexed family of* α*-cuts,*

(iii) $(\eta,\zeta) \in M^*(I)$ *iff* $\forall\alpha \in I_1$, $(\eta,\zeta)^{(\alpha)} \geq 0$.

Proposition 2.3 [7]. *For each fixed* $\alpha \in I_1$, *taking* α*-cuts is an order preserving real algebra homomorphism from* $M(I)$ *onto* R.

Definition 2.4. *A fuzzy semi-inner-product on a unitary* $M(I)$*-modulo* X *is a function* $\cdot : X \times X \to M(I)$ *which satisfies the following three axioms;*

(F_1) $\cdot$ *is linear in one component only.*

(F_2) $x \cdot x > \bar{0}$ *for every nonzero* $x \in X$.

(F_3) $|\lfloor x \cdot y\rfloor|^2 \leq (x \cdot x)(y \cdot y)$ *for every* $x, y \in X$

The pair $(X, \cdot)$ is called a *fuzzy semi-inner-product space.* The fuzzy pseudo-norm $\|\ \|$ associated with $(X,\cdot)$ is the function $\|\ \| : X \to R^*(I)$ defined for all $x \in X$ by $\|x\| = |\lfloor x \cdot x\rfloor|^{\frac{1}{2}}$ with values in $R^*(I)$. We write $(X, \cdot, \|\ \|)$ to show that the norm $\|\ \|$ is the function thus derived from the fuzzy semi-inner-product.

Let us new define the real quadratic form $<,>_\alpha$: $X \times X \to R$ for every $x, y \in X$,$\alpha \in I_1$ fixed, by $< x, y >_\alpha = (x \cdot y)^{(\alpha)}$.

Lemma 2.5. *If for $\alpha \in I_1$ and $x \in X, (x \cdot x)^{(\alpha)} = 0$, then $(x, y)^{(\alpha)} = 0$ for all $y \in X$.*

Proof. By (F_3), we obtain $| < x, y >_\alpha |^2 \leq < x, x >_\alpha < y, y >_\alpha$. This yields $((x \cdot y)^{(\alpha)})^2 \leq (x \cdot x)^{(\alpha)} (y \cdot y)^{(\alpha)}$. If $(x \cdot x)^{(\alpha)} = 0$ for some $\alpha \in I_1$, then $(x \cdot y)^{(\alpha)} = 0$ for all $y \in X$.

Proposition 2.6. *If $(\eta, \zeta) \geq (\xi, \lambda)$ in $M^*(I)$, then $|\lfloor(\eta\zeta)\rfloor| \geq |\lfloor(\xi, \lambda)\rfloor|$ in $R^*(I)$.*

Proof. . Since $(\eta, \zeta) \in M^*(I)$, then $\theta \geq (\zeta, \eta)$ for all $\theta \in R^*(I)$. From the properties of the infimum and Definition 1.8, we have

$$\begin{aligned} |\lfloor(\eta\zeta)\rfloor| &= inf\{\theta \in R^*(I) : \theta \geq (\eta, \zeta)\} \\ &\geq \inf\{\theta \in R^*(I) : \theta \geq (\xi, \lambda)\} \\ &= |\lfloor(\xi, \lambda)\rfloor|. \end{aligned}$$

In the following proposition, we will denote the elements of the ring $M(I)$ by just a single letter.

Proposition 2.7. *Let $x, y \in M^*(I)$ be such that $y^{(\alpha)} = 0$ whenever $\alpha \in I_1$ satisfies $x^{(\alpha)} = 0$. Then for all $z \in M^*(I)$ we have $xz \geq xy$ iff $z \geq y$.*

proof. $xz \geq xy$ iff $(xz)^{(\alpha)} \geq (x, y)^{(\alpha)}, \forall \alpha \in I_1$. Using the properties of α-cuts on $M(I)$, we have $x^{(\alpha)}$, $y^{(\alpha)}$,$z^{(\alpha)} \geq 0$, and $x^{(\alpha)} z^{(\alpha)} \geq x^{(\alpha)} y^{(\alpha)}$. If $x^{(\alpha)} \neq 0$, then $z^{(\alpha)} \geq y^{(\alpha)}$. If $x^{(\alpha)} = 0$, then $y^{(\alpha)} = 0$, and hence $z^{(\alpha)} \geq y^{(\alpha)}$. Since this holds for all $\alpha \in I_1$, then we conclude thar $z \geq y$.

Theorem 2.8. *Let $(x, \cdot \| \; \|)$ be a fuzzy semi-inner-product space. Then, considering X as a real vector space, $\| \; \|$ is indeed a fuzzy pseudo-norm on X. It also satisfies :*

(i) $\|x\| > \bar{0}$ *for every nonzero* $x \in X$, *and*

(ii) $\|(\eta, \zeta)x\| \leq |\lfloor(\eta,\zeta)\rfloor| \ |x\|$ *for all* $x \in X$ *and* $(\eta, \zeta) \in M(I)$.

Proof. $\|tx\|^2 = |\lfloor tx \cdot tx \rfloor| = t|\lfloor x \cdot tx \rfloor| \leq |t| \ |x\| \ |tx\|$. Thus $\|tx\| \leq |t| \ |x\|$. For $\lambda \neq 0$, $\|x\| = \|\frac{1}{t}tx\| \leq \frac{1}{|t|}\|tx\|$, $|t| \ |x\| \leq \|tx\|$. Therefore $\|tx\| = |t| \ |x\|$, $\forall x \in X$, $t \in R$. Also,

$$\begin{aligned} \|x+y\|^2 &= |\lfloor (x+y)\cdot(x+y)\rfloor| \\ &= |\lfloor x\cdot(x+y)\rfloor| + |\lfloor y\cdot(x+y)\rfloor| \\ &\leq \|x\|\|x+y\| + \|y\|\|x+y\| \\ &= (\|x\| + \|y\|)\|x+y\|. \end{aligned}$$

Then $\|x+y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$

Hence $\| \ \|$ is a fuzzy pseudo-norm on X.

To prove (i), let $x \in X$ be a nonzero element in X. Then $x \cdot x > \bar{0}$ in $M(I)$ and hence $\|x\| = |\lfloor x \cdot x \rfloor|^{\frac{1}{2}} > \bar{0}$ in $R^*(I)$.

To prove (ii), let $(\eta, \zeta) \in M(I)$, then

$$\begin{aligned} \|(\eta,\zeta)x\|^2 &= |\lfloor(\eta,\zeta)x \cdot (\eta,\zeta)x\rfloor| \\ &= |\lfloor(\eta,\zeta)^2(x,x)\rfloor|. \end{aligned}$$

But, since $(M(I), |\lfloor \ \rfloor|)$ is a fuzzy pseudo-normed algebra(Theorem 1.11(v)), then

$$\begin{aligned} |\lfloor(\eta,\zeta)^2(x\cdot x)\rfloor| &\leq |\lfloor(\eta,\zeta)^2\rfloor||\lfloor(x\cdot x)\rfloor| \\ &= |\lfloor(\eta,\zeta)\rfloor|^2|\lfloor(x\cdot x)\rfloor| \\ &= |\lfloor(\eta,\zeta)\rfloor|^2\|x\|^2, \end{aligned}$$

where the first equality is true by Proposition 1.13 (iv).

Due to the fact that the square roots exist and are unique in $R^*(I)$, we obtain by Proposition 1.13 (i),

$$\|\lfloor(\eta,\zeta)x\rfloor\| \leq \|\lfloor(\eta,\zeta)\rfloor\| \, \|x\|.$$

Proposition 2.9. *Let* $\{(X_i, \cdot_i, \| \ \|_i) : i = 1, 2, \cdots, n\}$ *be a finite indexed family of fuzzy semi-inner-product spaces, and let* $X = \prod X_i = \{(x_1, x_2, \cdots, x_n) : x_1 \in X_1, x_2 \in X_2, \cdots, x_n \in X_n\}$ *be the product module of the* X_i*'s(under coordinate-wise operations). Define the function* $\cdot : X \times X \to M(I)$ *by for* $x = (x_i)$ *and* $y = (y_i)$ *in* X,

$$x \cdot y = \bigoplus_{i=1}^{n} x_i \cdot y_i.$$

Then this function $\cdot$ *is a fuzzy semi-inner-product on* X.

Proof. The proof is straightforward from the properties of each fuzzy semi-inner-product $\cdot_i$ and the properties of fuzzy summation.

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