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RECURSIVE PROPERTIES OF A MAP ON THE CIRCLE

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1. Introduction

Let I be the interval, S^1 the circle and let X be a compact metric space. And let $C^\circ(X,X)$ denote the set of continuous maps from X into itself. For any $f \in C^\circ(X,X)$, let $P(f), R(f), \Gamma(f), \Lambda(f)$ and $\Omega(f)$ denote the collection of the periodic points, recurrent points, γ-limit points, ω-limit points and nonwandering points, respectively. Let $T(f) = \{x \in X | f \text{ is not a local homeomorphism at } x\}$ denote the set of turning points of f. A map $f \in C^\circ(X,X)$ is said to be piecewise monotone if the set $T(f)$ is finite. For any piecewise monotone f, we know that $\Lambda(f) = \overline{P(f)}$ by the result of Sarkovskii [4] and Nitecki [3]. Hence $\Lambda^2(f) = \Lambda(\overline{P(f)})$. On the other hand, J.C.Xiong [5] proved that for any piecewise monotone f, $\overline{P(f)} = \Lambda(\overline{P(f)})$. Therefore $\Lambda^2(f) = \Lambda(f)$ for any picewise monotone f. Also J.C.Xiong [5] proved that for a piecewise monotone f, $x \in \overline{P(f)}$ if and only if $\overline{P(f)} = \Lambda(\overline{P(f)})$. In this paper, we obtain the following similar results for maps of the circle :

Theorem A. *Let $f \in C^0(S^1,S^1)$. Suppose that $T(f)$ is finite. Then we have*

$$\Gamma(f) = \Lambda(f).$$

Theorem B. *Let $f \in C^0(S^1,S^1)$. Suppose that $T(f)$ is finite. Then we have*

$$\Lambda^2(f) = \Lambda(f).$$

2. Definitions and preliminaries

Let $f \in C^{\circ}(X, X)$. For $x \in X$, a point $y \in X$ is called an ω-limit point of x if there exists a sequence n_i of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to y$. Denote $\omega(x)$ the set of ω-limit points of x. A point $x \in X$ is called a recurrent point of f if $x \in \omega(x)$. A point $y \in X$ is called an α-limit point of x if there exists a sequence n_i of positive integers with $n_i \to \infty$ and a sequence y_i of points such that $f^{n_i}(y_i) = x$ and $y_i \to y$. The set of α-limit points of x denoted by $\alpha(x)$.

Let $\Lambda(f) = \cup_{x \in X} \omega(x)$. Let $\Lambda^0(f) = X$, and define, inductively $\Lambda^n(f) = \Lambda(\Lambda^{n-1}(f))$ for any $n \geq 1$. Obviously, $\Lambda^1(f) \supset \Lambda^2(f) \supset \Lambda^3(f) \supset \cdots$. The set $\Lambda^{\infty}(f) = \bigcap_{n=1}^{\infty} \Lambda^n(f)$ is called the attracting centre of f.

The forward orbit $O_P(x)$ of $x \in X$ is the set $\{f^k(x) | k = 0, 1, 2, \cdots\}$, and the reverse orbit $O_N(x)$ of $x \in X$ is the set $\bigcup_{n=1}^{\infty} f^{-n}(x)$. Usually the forward orbit of x is simply called the orbit of x.

3. Main results

The following lemmas and proposition found in [1].

Lemma 1. *Let $f \in C^{\circ}(S^1, S^1)$ and $x \in \Omega(f)$. Then we have $x \in \alpha(x)$.*

Lemma 2. *Let $f \in C^0(S^1, S^1)$ and $I = [a, b]$ be an arc for some $a, b \in S^1$ with $a \neq b$, and let $I \cap P(f) = \phi$.*

(a) Suppose that there exists $x \in I$ such that $f(x) \in I$ and $x < f(x)$. Then

(1) if $y \in I, f(y) \in I$, $x < y$ and $f(y) < y$, then $[x, y]$ f-covers $[f(x), b]$, and

(2) if $y \in I, f(y) \notin I$ and

(i) $y < x$, then $[y, x]$ f-covers $[f(x), f(y)]$.

(ii) $x < y$, then $[x, y]$ f-covers $[f(x), f(y)]$.

(b) Suppose that there exists $x \in I$ *such that* $f(x) \in I$ *and* $x > f(x)$*. Then*
(1) if $y \in I, f(y) \in I$, $y < x$ *and* $y < f(y)$*, then* $[x, y]$ f*-covers* $[a, f(x)]$*, and*
(2) if $y \in I, f(y) \notin I$ *and*
(i) $y < x$*, then* $[y, x]$ f*-covers* $[f(y), f(x)]$.
(ii) $x < y$*, then* $[x, y]$ f*-covers* $[f(y), f(x)]$.

Proposition 1. *Let* $f \in C^{\circ}(S^1, S^1)$*. Then we have*

$$P(f) \subset R(f) \subset \Gamma(f) \subset \overline{R(f)} \subset \Lambda(f) \subset \Omega(f).$$

Lemma 3. *Let* $f \in C^{\circ}(S^1, S^1)$*. If* $q \in T(f^n)$*, then there exists a point* $q' \in T(f)$ *such that* $q \in O_N(q')$ *and* $f^n(q) \in O_P(q')$.

Theorem A. *Let* $f \in C^0(S^1, S^1)$*. Suppose that* $T(f)$ *is finite. Then we have*

$$\Gamma(f) = \Lambda(f).$$

Proof. It suffices to show that $\Lambda(f) \subset \Gamma(f)$. Suppose that $x \in \Lambda(f) \setminus \Gamma(f)$. Then there exists an open arc (a, b) containing x such that $(a, b) \cap O_P(x) = \phi$ Without loss of generallty, we may assume that there exists $y \in S^1$ and $n_i \to \infty$ such that $f^{n_i}(y) \to x$ and $a < f^{n_1}(x) < \cdots < f^{n_i}(y) < x$. Let us take a sequence of points z_i with $x < \cdots < z_i < \cdots < z_1 < b$ such that $z_i \to x$. For each i, thete exist $j_i > i$ such that $(f^{n_{j_i}}(y), z_{j_i}) \cap O_N(f^{n_i}(y)) = \phi$. For if $(f^{n_{j_i}}(y), z_{j_i}) \cap O_N(f^{n_i}(y)) \neq \phi$ for all $j > i$, then $x \in \omega(f^{n_i}(y)) \cap \alpha(f^{n_i}(y)) \subset \Gamma(f)$, a contradiction. Assume $j_{i+1} > j_i$. Fixed $i > 0$. By Lemma 1, we can choose $u_i \in (f^{n_{j_i}}(y), z_{j_i})$ and an increasing sequence of positive integers m_i such that $f^{m_i}(u_i) = x$ By taking subsequence if we need, we have the following two cases:

Case I: $u_1 < u_2 < \cdots < x$

Since $f^{n_{i-1}}(y) \notin f^{m_i}([u_{i-1}, x])$, by Lemma 2,

$$[u_{i-1}, u_i]\ f^{m_i} - \text{covers } [x, f^{m_i}(u_{i-1})]$$

and

$$[u_i, x]\ f^{m_i} - \text{covers } [x, f^{m_i}(x)].$$

In particular, both $[u_{i-1}, u_i]$ and $[u_i, x]$ f^{m_i} – covers $[x, b]$. Therefore, if we take an arbtrary point $c \in (x, b)$, then there exist $d \in (u_{i-1}, u_i)$ and $e \in (u_i, x)$ such that $f^{m_i}(d) = f^{m_i}(e) = c$. Hence there exists a turning points q_i of f^{m_i} with $u_{i-1} < q_i < x$.

Case II. $x < \cdots < u_i < \cdots < u_2 < u_1$

We know that $f^{n_{i-1}}(y) \notin f^{m_i}([x, u_{i-1}])$. By Lemma 2,

$$[u_i, u_{i-1}]\ f^{m_i} - \text{covers } [f^{m_i}(u_{i-1}), x]$$

and

$$[x, u_i]\ f^{m_i} - \text{covers } [f^{m_i}(x), x].$$

In particular, both $[u_i, u_{i-1}]$ and $[x, u_i]$ f^{m_i} – covers $[a, x]$. By the same way in the Case I, we have a turning points q_i' of f^{m_i} with $x < q_i' < u_{i-1}$.

In any cases, we can have a turning point $q_i \in (u_{i-1}, z_i)$ of f^{m_i} By lemma 3, there exists $c_i \in T(f)$ such that $q_i \in O_N(c_i)$ and $f^{m_i}(q_i) \in O_P(c_i)$. Hence $(f^{n_i}(y), z_i) \cap O_N(c_i) \neq \phi$ and $(f^{n_i}(y), z_i) \cap O_P(c_i) \neq \phi$ for all $i \geq 1$.

Since $T(f)$ is finite, the sequence c_i has a subsequence taking by a constant value, say, c_k. In this case, $x \in \omega(c_k) \cap \alpha(c_k) \subset \Gamma(f)$. This is a contradiction.

The proof of theorem is complete.

The following corollary is immediate consequence of Proposition 1 and Thereom A

Corollary 1. *Let $f \in C^0(S^1, S^1)$. Suppose that $T(f)$ is finite. Then the followings are equivalent:*

(1) Every γ-limit point is recurrent.

(2) Recurrent points forms a closed set.

(1) Every ω-limit point is recurrent.

The following proposition found in [2].

Proposition 2. *Let $f \in C^0(S^1, S^1)$. Then we have*

$$\Lambda^\infty(f) = \cdots = \Lambda^2(f) = \Lambda(\Omega(f)) = \Lambda(\overline{R(f)}) = \Lambda(\Gamma(f)) = \Gamma(f)).$$

Theorem B. *Let $f \in C^0(S^1, S^1)$. Suppose that $T(f)$ is finite. Then we have*

$$\Lambda^2(f) = \Lambda(f).$$

Proof. We know that $\Lambda(f) = \overline{R(f)} = \Gamma(f)$ by Theorem A. By Proposition 2, $\Lambda^2(f) = \Gamma(f) = \Lambda(f)$.

The proof is complete.

Corollary 2. *Let $f \in C^0(S^1, S^1)$. Suppose that $T(f)$ is finite. Then we have*

$$\begin{aligned}\Lambda^\infty(f) &= \cdots = \Lambda^2(f) = \Lambda(f) \\ &= \Lambda(\Omega(f)) = \Lambda(\overline{R(f)}) = \Lambda(\Gamma(f)) \\ &= \overline{R(f)} = \Gamma(f).\end{aligned}$$

Therefore the attracting centre of f is $\Lambda(f)$, and $\Lambda(\overline{R(f)}) = \overline{R(f)}$.

Proof. By Proposition 2,

$$\Lambda^\infty(f) = \cdots = \Lambda^2(f) = \Lambda(\Omega(f)) = \Lambda(\overline{R(f)}) = \Lambda(\Gamma(f)) = \Gamma(f)).$$

We know that $\Lambda(f) = \Gamma(f)$ by Theorem B. Therefore

$$\begin{aligned}\Lambda^{\infty}(f) &= \cdots = \Lambda^2(f) = \Lambda(f) \\ &= \Lambda(\Omega(f)) = \Lambda(\overline{R(f)}) = \Lambda(\Gamma(f)) \\ &= \overline{R(f)} = \Lambda(f).\end{aligned}$$

The proof is complete.

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