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LIMIT SETS AND PROLONGATIONAL LIMIT SETS IN DYNAMICAL POLYSYSTEMS

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1. Introduction

In stability theory of polysystems two concepts that play a very important role are the limit set and the prolongational limit set.

For the above two concepts, A.Bacciotti and N.Kalouptsidis studied their properties in a locally compact metric space [2]. In this paper we investigate their results in c-first countable space which is more a general space than a metric space.

Let X be a locally compact c-first countable space unless otherwise stated, $\mathbb{R}^+$ the set of nonnegative real numbers and 2^X the set of all subsets of X. A dynamical system on X is a continuous map $\pi : X \times \mathbb{R} \to X$ with the following properties :

(a) $\pi(x,0) = x$ for all $x \in X$

(b) $\pi(\pi(x,s),t) = \pi(x,s+t)$ for all $x \in X$ and $s,t \in \mathbb{R}$.

We call a family of dynamical systems $\{\pi_i | i \in I\}$ a dynamical polysystem on X.

2. Limit Sets

The purpose of this section is to introduce the concept of the limit set and extend some of their properties stated in [2] to c-first countable space.

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Definition 2.1. For a polysystem $\{\pi_i | i \in I\}$ with reachable sets $R(x,t)$ its limit set at a point x is defined as

$\Lambda(x) = \{y \in X|$ there exist sequences $t_n \to +\infty, y_n \to y$ such that $y_n \in R(x,t_n)\}$.

The next proposition provides alternate description of the limit set.

Proposition 2.2. *For any x in X, $\Lambda(x) = \bigcap_{t\in\mathbb{R}^+} \overline{R(x,[t,\infty))}$.*

Proof. Let $y \in \Lambda(x)$. Then there exist sequences $t_n \to +\infty, y_n \to y$ such that $y_n \in R(x,t_n)$. For each $t \in \mathbb{R}^+$, since $t_n \to +\infty$, we may assume that $t_n \geq t$ for all n. Thus we have $y_n \in R(x,t_n) \subset R(x,[t,\infty))$. This shows that $y \in \overline{R(x,[t,\infty))}$. Since t is arbitrary, $y \in \bigcap_{t\in\mathbb{R}^+} \overline{R(x,[t,\infty))}$.

Conversely, let $y \in \bigcap_{t\in\mathbb{R}^+} \overline{R(x,[t,\infty))}$. We can choose a countable basis (U_n) at y with $U_{n+1} \subset U_n$. For all integers n, since $y \in \overline{R(x,[n,\infty))}, U_n \cap R(x,[n,\infty)) \neq \phi$. Therefore there exists $y_n \in U_n \cap R(x,[n,\infty))$ with $y_n \to y$. Also, there exists $t_n \geq n$ such that $y_n \in R(x,t_n)$. It is clear that $t_n \to +\infty$. Hence $y \in \Lambda(x)$ and the proposition is proved.

A following corollary follows from the above proposition and the finite intersection property.

Corollary 2.3. *For all $x \in X$,*

(1) *$\Lambda(x)$ is closed and positively invariant.*
(2) *Let $\overline{R(x)}$ be a compact subset of X.*

Then $\Lambda(x)$ is nonempty.

In a dynamical system π, the limit set is related to the closure of the positive orbit in the following way

$$\overline{\gamma^+(x)} = \gamma^+(x) \cup \Lambda^+(x)$$

In order to state an analogous relation for dynamical polysystems, we consider the following definiton.

Definition 2.4. $R^*(x,t) = \{y \in X \mid$ there are sequences $y_n \to y, t_n \to t$ such that $y_n \in R(x,t_n)\}$.

Propersition 2.5. $\overline{R(x)} = R^*(x,\mathbb{R}^+) \cup \Lambda(x)$.

Proof. Let $y \in \overline{R(x)}$. Then there is a sequence $y_n \in R(x)$ such that $y_n \to y$. Therefore, there is a sequence $t_n \in \mathbb{R}^+$ such that $y_n \in R(x,t_n)$. We may assume that either $t_n \to t \in \mathbb{R}^+$, or $t_n \to +\infty$. In the first case $y \in R^*(x,t)$ by definition. In the second case $y \in \Lambda(x)$ by definition. Hence $y \in R^*(x,\mathbb{R}^+) \cup \Lambda(x)$.

Conversely, we prove that $\overline{R(x)} \supset R^*(x,\mathbb{R}^+) \cup \Lambda(x)$. $\Lambda(x) \subset \overline{R(x)}$ holds always. For any $t \in \mathbb{R}^+$,

$$R^*(x,t) \subset \overline{R(x,[0,s])} \subset \overline{R(x,\mathbb{R}^+)} = \overline{R(x)} \text{ for } s > t.$$

Thus we have $\overline{R(x)} \supset R^*(x,\mathbb{R}^+) \cup \Lambda(x)$ and the proposition is completed.

The following proposition is useful in the study of attractivity properties.

Proposition 2.6. *Let $t \in \mathbb{R}^+, x \in X$ and $y \in R^*(x,t)$. Then$\Lambda(y) \subset \Lambda(x)$*

Proof. To avoid cumbersome notation, we write the expression $w = \pi_{i_k}(\cdots, \pi_{i_1}(x,t_1),\cdots,t_k) \in R(x,t), \sum_{i=1}^k t_i = t$ as $w = \pi(i_1,\cdots,i_k,x,t_1,\cdots,t_k)$. By assumption, there are sequences $y_m \to y, t_m \to t$ such that $y_m \in R(x,t_m)$. Let $z \in \Lambda(y)$. Then there are sequences $z_n \to z, s_n \to +\infty$ such that $z_n \in R(y,s_n)$. For any integer n, there are $i_1^n,\cdots,i_{k_n}^n \in I$ and $r_1^n,\cdots,r_{k_n}^n \in \mathbb{R}^+$ such that $\sum_{j=1}^{k_n} r_j^n = s_n, z_n = \pi(i_1^n,\cdots,i_{k_n}^n,y,r_1^n,\cdots,r_{k_n}^n)$. Let (U_n) be a basis at z with $U_n \supset U_{n+1}$. Then there is an integer n_1 such that $z_{n_1} \in U_1$. Also, there is an integer m_1 such that $w_1 \equiv \pi(i_1^{n_1},\cdots,i_{k_{n_1}}^{n_1},y_{m_1},r_1^{n_1},\cdots,r_{k_{n_1}}^{n_1}) \in U_1$. We can choose an integer $n_2 > n_1$ so that $z_{n_2} \in U_2$. Thus there is an integer $m_2 > m_1$, such that $w_2 \equiv \pi(i_1^{n_2}, \cdots, i_{k_{n_2}}^{n_2}, y_{m_2}, r_1^{n_2}, \cdots, r_{k_{n_2}}^{n_2}) \in U_2$. Continuing this process, the resulting sequence w_j converges to z. Here, $w_j \in R(y_{m_j}, s_{n_j}) \subset R(R(x, t_{m_j}), s_{n_j}) = R(x, t_{m_j} + s_{n_j})$. Since $t_{m_j} + s_{n_j} \to +\infty, z \in \Lambda(x)$. This completes the proof.

The next theorem indicates that the positive orbit of a point x is attracted $\Lambda(x)$ if compactness is assumed.

Theorem 2.7. *If a limit set $\Lambda(x)$ is nonempty and compact, then for any neighborhood U of $\Lambda(x)$, there is a $t \in \mathbb{R}^+$ such that $R(x, [t, \infty)) \subset U$.*

Proof. Suppose that the conclusion is not true. Then there is a neighborhood U of $\Lambda(x)$ such that for all $t \in \mathbb{R}^+, R(x, [t, \infty)) \not\subset U$. Choose a neighborhood V of $\Lambda(x)$ so that $\overline{V}$ is compact and $\overline{V} \subset U$. For all $t \in \mathbb{R}^+, R(x, [t, \infty)) \not\subset \overline{V}$. Let $y \in \Lambda(x)$. Since $\Lambda(x) \subset \overline{R(x, [n, \infty))}$ and V is a neighborhood of y, there exists $t_n \geq n$ such that $R(x, t_n) \cap V \neq \phi$. Also, there exists $s_n > t_n$ such that $R(x, s_n) \cap (X - \overline{V}) \neq \phi$. Since $R(x, [t_n, s_n])$is connected, there is a sequence $r_n \in [t_n, s_n]$ such that $R(x, r_n) \cap \partial\overline{V} \neq \phi$. Let $z_n \in R(x, r_n) \cap \partial\overline{V}$. Since $\partial\overline{V}$ is compact, there exists a sequence $z_n \to z \in \partial\overline{V}$. Since $r_n \to +\infty, z \in \Lambda(x)$. This is a contradiction. Thus the theorem is proved.

3. Prolongational Limit Sets

We introduce the concept of the prolongational limit set and extend some of its basic properties stated in [2] to c-first countable space.

Definition 3.1. Let x in X. The prolongational limit set of x is a subset $J(x) = \bigcap_{U \in N(x), t \in \mathbb{R}^+} \overline{R(U, [t, \infty))}$, where $N(x)$ is a family of all neighborhoods of x.

It is obvious that $J(x)$ is closed, positively invariant and $\Lambda(x) \subset J(x)$.

The following proposition indicates alternate description of the prolongational limit set.

Proposition 3.2. *$J(x) = \{y \in X \mid$ there are sequences $x_n \to x$, $y_n \to y, t_n \to +\infty$ such that $y_n \in R(x_n, t_n)\}$.*

Proof. Let $y \in J(x)$. Take a basis at x and y, (U_n) and (V_n), respectively, with $U_n \supset U_{n+1}$, $V_n \supset V_{n+1}$. Then from definition of $J(x)$, for any integer n, we have $R(U_n, [n, \infty)) \cap V_n \neq \phi$. Thus there are sequences $x_n \in U_n, y_n \in V_n, t_n \geq n$ such that $y_n \in R(x_n, t_n)$. Therefore, $x_n \to x, y_n \to y$ and $t_n \to +\infty$.

Conversely, suppose that there are sequences $x_n \to x, y_n \to y$ and $t_n \to +\infty$ such that $y_n \in R(x_n, t_n)$. For any neighborhoods U, V of x and y, respectively, and $t \in \mathbb{R}^+$, there exists an integer m such that $x_m \in U, y_m \in V, t_m \geq t$. Thus we have $R(U, [t, \infty)) \cap V \neq \phi$ and so $y \in \overline{R(U, [t, \infty))}$. Since U is any neighborhood of x, $y \in J(x)$. Hence the proposition is proved.

In order to state the relation between the prolongation set and the prolongational limit set, we consider the multivalued map $DR : X \times \mathbb{R}^+ \to 2^X$ defined by the following definition.

Definition 3.3. For each x in X and t in $\mathbb{R}^+$, $DR(x, t) = \{y \in X \mid$ there exist sequences $x_n \to x, y_n \to y$ and $t_n \to t$ such that $y_n \in R(x_n, t_n)\}$.

We let $DR(x, \mathbb{R}^+) = \bigcup_{t \in \mathbb{R}^+} DR(x, t)$.

Proposition 3.4. *For each $x \in X$ it holds*

$$DR(x) = DR(x, \mathbb{R}^+) \cup J(x).$$

Proof. Let $y \in DR(x)$. Then there exist sequences $x_n \to x, y_n \to y$ and $t_n \in \mathbb{R}^+$ such that $y_n \in R(x_n)$. We may assume that either $t_n \to t \in \mathbb{R}^+$ or $t_n \to +\infty$. In the first case, we have $y \in DR(x,t)$ by definition and so $y \in DR(x,\mathbb{R}^+)$. In the second case, $y \in J(x)$ by definition. Thus $DR(x) \subset DR(x,\mathbb{R}^+) \cup J(x)$.

Next we will prove $DR(x) \supset DR(x,\mathbb{R}^+) \cup J(x)$. Let $y \in DR(x,\mathbb{R}^+) \cup J(x)$. If $y \in DR(x,\mathbb{R}^+)$, then there is $t \in \mathbb{R}^+$ such that $y \in DR(x,t)$. Thus there are sequences $x_n \to x, y_n \to y$ and $t_n \to t$ such that $y_n \in R(x_n,t_n)$. By definition, $y \in DR(x)$. If $y \in J(x)$, then there are sequences $x_n \to x, y_n \to y$ and $t_n \to +\infty$ such that $y_n \in R(x_n,t_n)$. Clearly, $y \in DR(x)$. The converse is proved. Hence the propositon is complete.

Proposition 3.5. *Let $J(x)$ be nonempty and compact. Then $\Lambda(x)$ is nonempty and compact.*

Proof. Suppose that $\Lambda(x)$ is empty. We claim that $\overline{R(x)} \cap J(x) = \phi$. Let $\overline{R(x)} \bigcap J(x) \neq \phi$. Then there is $y \in \overline{R(x)} \cap J(x)$. By proposition 2.5, we have $\overline{R(x)} = R^*(x,\mathbb{R}^+)$ and so $y \in R^*(x,t)$ for some $t \in \mathbb{R}^+$. By propositon 2.6, $\Lambda(y) = \phi$. Since $J(x)$ is closed, positively invariant and $y \in J(x), \overline{R(y)} \subset J(x)$. From compactness of $J(x), \overline{R(y)}$ is compact. By corollary 2.3, $\Lambda(y) \neq \phi$. This is a contradiction. Thus it follows that $\overline{R(x)} \bigcap J(x) = \phi$.

Let us show that there exists a neighborhood U of $J(x)$ such that $\overline{U}$ is compact and $\overline{U} \cap R(x) = \phi$. For all $y \in J(x), y \notin \overline{R(x)}$. Thus there exists a neighborhood V_y of y such that $V_y \cap R(x) = \phi$. We choose a neighborhood U_y of y so that $\overline{U_y}$ is compact and $\overline{U_y} \subset V_y$. A family $\{U_y | y \in J(x)\}$ is and open cover of $J(x)$. Since $J(x)$ is compact, there is a finite subconver $\{U_{y_i} | y_i \in J(x), i = 1,2,\cdots,n\}$. It follows that $J(x) \subset \cup_{i=1}^n U_{y_i} \subset \cup_{i=1}^n \overline{U_{y_i}} = \cup_{i=1}^n \overline{U_{y_i}} \subset \cup_{i=1}^n V_{y_i}$. Set $\cup_{i=1}^n U_{y_i}$ by U. Then $\overline{U}$ is compact and $\overline{U} \cap R(x) = \phi$. Fix i and take $t_n \to +\infty$. We have $\pi_i(x,t_n) \in R(x)$ and so $\pi_i(x,t_n) \in X - \overline{U}$. By continuity of π_i, there is a nighborhood $V_n \subset U_n$ of x such that $\pi_i(V_n,t_n) \subset X - \overline{U}$, where (U_n) is a basis at x with $U_n \supset U_{n+1}$. For each n , there exists a sequence $s_n > t_n$ and $x_n \in V_n$ such that $U \cap R(x_n,s_n) \neq \phi$. Since $\pi_i(x_n,t_n) \in \pi_i(V_n,t_n) \subset X - \overline{U}$ and $\pi_i(x_n,t_n) \in R(x_n,t_n)$, we have $R(x_n,t_n) \cap (X - \overline{U}) \neq \phi$. From connectedness of $R(x_n,[t_n,s_n])$, there is a sequence $r_n \in [t_n, s_n]$ such that $R(x_n, t_n) \cap \partial\overline{U} \neq \phi$. We choose $z_n \in R(x_n, t_n) \cap \partial\overline{U}$. Since $\partial\overline{U}$ is compact, there is a sequence $z_n \to z \in \partial\overline{U}$. It is obvious that $x_n \to x$ and $r_n \to +\infty$. Therefore, $z \in J(x)$. This contradicts the fact that $z \notin J(x)$. Thus $\Lambda(x)$ is nonempty. That $\Lambda(x)$ is compact is clear. Hence the proposition is completed.

The next theorem states that a nonempty compact prolongational limit set uniformly attracts its positive orbit.

Theorem 3.6. *Suppose that $J(x)$ is nonempty and compact. Then for any neighborhood U of $J(x)$, there is a neighborhood V of x and $t \in \mathbb{R}^+$ such that $R(V, [t, \infty)) \subset U$.*

Proof. Suppose the conclusion is not true. Then there exists a neighborhood U of $J(x)$ such that for any neighborhood V of x and $t \in \mathbb{R}^+, R(V, [t, \infty)) \not\subset U$. We choose a neighborhood W fo $J(x)$ so that $\overline{W}$ is compact and $\overline{W} \subset U$. Since $\Lambda(x)$ is nonempty and compact and W is a neighborhood of $\Lambda(x)$, by theorem 2.7 there exists $t \in \mathbb{R}^+$ such that $R(x, [t, \infty)) \subset W$. Let $t_n \to +\infty$ with $t_n \geq t$. Fix i. We have $\pi_i(x, t_n) \in R(x, t_n) \subset R(x, [t, \infty)) \subset W$. By continuity of π_i, there is a neighborhood $V_n \subset U_n$ of x such that $\pi_i(V_n, t_n) \subset W$, where (U_n) is a basis at x with $U_n \supset U_{n+1}$. Thus there is a sequence $s_n > t_n$ such that $R(V_n, s_n) \not\subset \overline{W}$. Let $x_n \in V_n$ with $R(x_n, s_n) \not\subset \overline{W}$. Since $\pi_i(x_n, t_n) \in \pi_i(V_n, t_n) \subset W$ and $\pi_i(x_n, t_n) \in R(x_n, t_n)$, we have $R(x_n, t_n) \cap W \neq \phi$. By connecdtedness of $R(x_n[t_n, s_n])$, there is a sequence $r_n \in [t_n, s_n]$ such that $R(x_n, r_n) \cap \partial\overline{W} \neq \phi$. Let $z_n \in R(x_n, r_n) \cap \partial\overline{W}$. Since $\partial\overline{W}$ is compact, there exists a sequence $z_n \to z \in \partial\overline{W}$. We clearly have $x_n \to x$ and $r_n \to +\infty$. Thus $z \in J(x)$. This is a contradiction. Hence the theorem is proved.

References

1. N. P. Bhatia and G. P. Szegö, *Stability Theory of Dynamical Systems*, Springer-Verlag, New York, 1970.

2. A. Bacciotti and N. Kalouptsidis, *Topological Dynamics of Control Systems: Stability and Attraction*, Nonlinear Analysis **10** (1986), 547-565.
3. J. Tsinias, *A Lyapunov description of stability in control systems*, Nonlinear Analysis **13** (1989), 63-74.
4. Y. H. Gu, *Stability of Dynamical Polysystems*, J. Chung Cheong Math. Soc. **7** (1994), 109-115.
5. J. S. Park, *Stability in topological dynamics*, J. Korean Math. Soc. **25(1)** (1988), 67-76.

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