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PROPER RATIONAL MAP IN THE PLANE

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In [6], the author studied the property of the Szegő kernel and had a result that if Ω is a smoothly bounded domain in $\mathbb{C}$ and the Szegő kernel associated with Ω is rational, then any proper holomorphic map from Ω to the unit disc U is rational. It leads to the study of the proper rational map of Ω to U. In this note, first we simplify the proof of the above result and prove an existence theorem of a proper rational map. Before we proceed to state our result, we must recall some preliminary facts.

Let $L^2(b\Omega) = \{\phi : \int_{b\Omega} |\phi|^2 < \infty\}$ and $H^2(b\Omega)$ be the closure of $A^\infty(b\Omega)$ in $L^2(b\Omega)$ where $A^\infty(b\Omega)$ is the functions on $b\Omega$ which are boundary values of functions in $A^\infty(\Omega)$.

The unique orthogonal projection $S : L^2(b\Omega) \to H^2(b\Omega)$ is called the Szegő projection and represented by the Szegő kernel $S(z,w)$ via

$$S\phi(z) = \int_{b\Omega} S(z,w)\phi(w)ds_w$$

for $\phi \in L^2(b\Omega)$ and $z \in \Omega$ (see [2; p.22]).

A continuous map $f : \Omega_1 \to \Omega_2$ between domains is called proper if$f^{-1}(K)$ is a compact subset of Ω_1 whenever K is a compact subset of Ω_2. The compactness of $f^{-1}(K)$ for every compact $K \subset \Omega_2$ is equivalent to the following requirement: If $\{z_k\}$ is a sequence in Ω_1 that tends to $b\Omega_1$, then the sequence $\{f(z_k)\}$ tends to $b\Omega_2$. Also, we note that if f extends continuously up to the boundary, then the

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condition that f be proper is equivalent to the condition that $f(b\Omega_1) \subset b\Omega_2$. It is easy to check that biholomorphic mappings are proper.

Suppose Ω_1 and Ω_2 are domains in $\mathbb{C}$ and $f : \Omega_1 \to \Omega_2$ is proper holomorphic. Let $\#(w)$ denote the number of points in the set $f^{-1}(w)$ for $w \in \Omega_2$. Then there is an integer m (the multiplicity of f) such that

$$\begin{aligned} \#(w) = m \quad & \text{for every regular value of } f \\ \#(w) < m \quad & \text{for every critical value of } f \end{aligned}$$

(see [7; p.303]).

A rational function f is defined to be a quotient of two polynomials P and Q such that $f = P/Q$. The finite Blaschke products

$$B(z) = c \prod_{i=1}^{m} \frac{\alpha_i - z}{1 - \overline{\alpha}_i z} \qquad (|\alpha_i| < 1, |c| = 1)$$

are examples of proper holomorphic maps of the unit disc U in $\mathbb{C}$ onto itself. We note that these maps are all possible proper holomorphic self-maps of the unit disc U. Indeed, if f is any proper holomorphic self-map of U, then $f \in C^\infty(\overline{U})$ (see [2; p.65]). So, by the definition of proper map, f maps the boundary bU of U to bU and $f^{-1}(0)$ is compact in U. Note that $f^{-1}(0)$ is a discrete finite set in U. Let $f^{-1}(0) = \{\alpha_1, \ldots, \alpha_n\}$ where each $\alpha_i \in U$. Denote $P(z) = \prod_{i=1}^{m}(\alpha_i - z)/(1 - \overline{\alpha}_i z)$. Then, P is an automorphism of U and maps bU onto bU. It is enough to show that $f = cP$ where $|c| = 1$. Since f/P and P/f has removable singularities, they are holomorphic in U. By the maximum principle, $|f/P|$ and $|P/f|$ have the maximum 1 on bU. It implies that $|f/P| = 1$ in U. Hence, $|f| = |P|$ leads to the desired conclusion.

Thus the proper maps of the disc in $\mathbb{C}$ onto itself are easy to classify. They are all rational functions that extend holomorphically past the disc. On the other hand, Alexander [1] proved that proper maps from the unit ball B_n in $\mathbb{C}^n$ to itself are necessarily automorphisms when $n \geq 2$. D'Angelo [4] classified all proper polynomial maps between balls. Now, we concentrate on proper maps from bounded domains in $\mathbb{C}$ to the unit disc U in $\mathbb{C}$.

Lemma 1. *Let Ω_1 be a smoothly bounded, n-connected domain in $\mathbb{C}$ and Ω_2 be a smoothly bounded, simply-connected domain in $\mathbb{C}$. Suppose $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping of multiplicity m. Let $F_1, \ldots ,F_m$ represent f^{-1} locally. Then, the Szegő kernels transform according to*

$$\sum_{i=1}^{m} S_1(z, F_i(w))^2 \overline{F_i'(w)} = f'(z) S_2(f(z), w)^2$$

for all $z \in \Omega_1$ and $w \in \Omega_2$.

Proof. See [6]. □

Theorem 2. *Let Ω_1 be a smoothly bounded domain in $\mathbb{C}$ whose associated Szegő kernel is rational and Ω_2 be the unit disc in $\mathbb{C}$. Then any proper holomorphic mapping $f : \Omega_1 \to \Omega_2$ is rational.*

Proof. Let m be the order of f and $F_1, \ldots ,F_m$ denote the m local inverses to f. By Lemma 1,

$$\sum_{k=1}^{m} S_1(z, F_i(w))^2 \overline{F_i'(w)} = f'(z) S_2(f(z), w)^2.$$

Since $S_2(f(z), w) = 1/2\pi(1 - f(z)\bar{w})$,

$$\sum_{k=1}^{m} S_1(z, F_i(w))^2 \overline{F_i'(w)} = \frac{f'(z)}{4\pi^2(1 - f(z)\bar{w})^2}. \tag{1}$$

By setting $w = 0$ in (1),

$$\sum_{i=1}^{m} S_1(z, F_i(0))^2 \overline{F_i'(0)} = f'(z)/4\pi^2$$

and hence $f'(z)$ is a rational function of z. By differentiating the transformation formula (1) for the Szegő kernels with respect to $\bar{w}$ and setting $w = 0$,

$$\sum_{i=1}^{m} 2S_1(z, F_i(0)) \overline{F_i'(0)}^2 + S_1(z, F_i(0))^2 \overline{F''(w)} = \frac{1}{2\pi^2} f'(z) f(z).$$

Since $S_1(\cdot,\cdot)$ is rational, $f'(z)f(z)$ is rational. Hence $f(z) = f(z)f'(z)/f(z)'$ is rational. □

Remark. Recently Bell [3] used this result to prove the Szegő kernel associated to the multiply connected domain is not rational.

Theorem 3. *For some smoothly bounded domain Ω in $\mathbb{C}$, there exists a proper rational map of Ω onto the unit disc U.*

Proof. Take a polynomial $p(z)$ with distinct zeros $z_1, \ldots, z_n$. Take a sufficiently small $\epsilon > 0$ so that the set $\{z \in \mathbb{C} : |p(z)| = \epsilon\}$ has n disjoint components $\{C_i\}_{i=1}^n$ where C_i is smooth, non-intersecting simple closed curve surrounding z_i, respectively. Without loss of generality, let $z_1 = 0$ and $\epsilon = 1$. Let $\Omega = \{z \in \mathbb{C} : |p(1/z)| > 1\}$. It is a smoothly bounded n-connected domain in $\mathbb{C}$ with $b\Omega = \cup_{i=1}^n \widetilde{C}_i$ where $1/z$ maps C_i to $\widetilde{C}_i$ and vice versa for $1 \le i \le n$. Here $\widetilde{C}_1$ is the outer boundary of Ω. The map $f(z) = 1/p(1/z)$ is a proper rational map of Ω onto U since f maps $b\Omega$ onto bU. □

We conjecture that for given smoothly bounded multiply-connected domain Ω in $\mathbb{C}$, there exists a proper holomorphic map of Ω onto the unit disc U which is not rational. The following theorem of Grunsky [5; p.133] may be helpful for the proof:

Let Ω denote a smoothly bounded n-connected domain in $\mathbb{C}$ with boundary $b\Omega$ and let $z_i \in C_i$ for $1 \le i \le n$ where $b\Omega = \cup_{i=1}^n C_i$. Then there exists a proper holomorphic map f of Ω onto the right half plane with multiplicity n and $f(z) \to \infty$ for $z \to z_i$. This function is unique up to a positive multiplicative and an imaginary additive constant.

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