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A NOTE ON QUASI-SIMILAR QUASI-HYPONORMAL OPERATORS

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1. Introduction

Let H be an arbitrary complex Hilbert space and let $\mathcal{L}(H)$ be the $*$-algebra of all bounded linear operators on H.

An operator T in $\mathcal{L}(H)$ is called normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and quasi-hyponormal if $T^*(T^*T - TT^*)A \geq 0$, or equivalently $\|T^*Tx\| \leq \|TTx\|$ for all x in H.

Then we have the following proper inclusion relation for these classes of operators([4]) ;

$$\text{Normal} \subset \text{Hyponormal} \subset \text{Quasi-hyponormal}.$$

Throughout this paper, $\mathcal{L}(H, K)$ denote the set of all bounded linear operators from a Hilbert space H into a Hilbert space K.

An operator X in $\mathcal{L}(H, K)$ is said to be quasi-invertible if X has zero kernel and dense range. Two operators S and T in $\mathcal{L}(H)$ and $\mathcal{L}(K)$ respectively are quasi -similar if there are quasi-invertible operators X and Y in $\mathcal{L}(H, K)$ and $\mathcal{L}(K, H)$ respectively which satisfy the equations $XS = TX$ and $YT = SY$.

Quasi-similarity was first introduced by B.Sz-Nagy and C.Foias in their theory of an infinite dimensional analogue of the Jordan form for certain classes of operators and it replaces the familiar notion of similarity which is the appropriate equivalence relation to use with finite dimensional Jordan forms.([3]).

Recall that an operator T is said to be Fredholm operator if both $\ker(T)$ and $\ker(T^*)$ are finite dimensional and the range of T is closed, where $\ker(T)$ is the

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX kernel of T. If T is a Fredholm operator, let $\text{ind}(T) = \dim(\ker(T)) - \dim(\ker(T^*))$ denote the index of T

In [1], Clary showed that if an operator S in $\mathcal{L}(H)$ is invertible, an operator T in $\mathcal{L}(K)$ is hyponormal, and an operator X in $\mathcal{L}(H, K)$ has dense range and satisfies $XS = TX$, then T is invertible. And in [5], L.R.Williams the following result;

Suppose that S and T are quasi-similar hyponormal operators. Then S is a Fredholm operator satisfying $\text{ind}(S) = 0$ if and only if T is a Fredholm operator satisfying $\text{ind}(T) = 0$.

As the results of this paper, if the hyponormal operators in above statements are replaced by the quasi-hyponormal operators, then we can obtained the same result.

2. The Main Theorem

A closed subspace M of a Hilbert space is said to be an invariant subspace for T in $\mathcal{L}(H)$ if $TM \subseteq M$.

the following result is due to N.C. Shah and I.H.Sheth.

Theorem 2.1([4]). *If T is quasi-hyponormal and is invariant under the subspace M of a Hilbert space H, then $T|_M$ is quasi-hyponormal.*

We prove some results which are extensions of known results for hyponormal operators.

Lemma 2.2. *Let T in $\mathcal{L}(H)$ be quasi-hyponormal and let $\{g_n\}_{n=0}^{\infty}$ be a sequence in H such that $Tg_{n+1} = g_n$ for all $n \geq 0$. Then either $\|g_0\| \geq \|g_1\| \geq \|g_2\| \geq \cdots$ or $\|g_n\| \to \infty$ as $n \to \infty$.*

Proof. For any g in H,

$$\begin{aligned}\|Tg\| &= < Tg, Tg >^{\frac{1}{2}} = < T^*Tg, g >^{\frac{1}{2}} \leq (\|T^*Tg\|\|T\|)^{\frac{1}{2}} \\ &\leq (\|T^2g\|\|g\|)^{\frac{1}{2}} \leq \frac{1}{2}(\|T^2g\| + \|g\|).\end{aligned}$$

Letting $g = g_{n+2}$, we see that $\|g_{n+1}\| \leq \frac{1}{2}(\|g_n\| + \|g_{n+2}\|)$, so the sequence $\{\|g_n\|\}$ is convex and the conclusion follows at onces.([6]).

Lemma 2.3. *Suppose S in $\mathcal{L}(H)$ is invertible, and T in $\mathcal{L}(K)$ is a quasi-hyponormal, and X in $\mathcal{L}(H,K)$ satisfies $XS = TX$. Then*

$$\|XS^{-1}h\| \le \|S^{-1}\|\|Xh\| \quad \textit{for all } h \in H.$$

Proof. Assume, without loss of generality, that $\dim H \ge 1$, and let $c = \|S^{-1}\| > 0$. Fix h in H and define $g_n = c^{-n}XS^{-n}h$ for $n \ge 0$. Then $cTg_{n+1} = g_n$ and $\|g_n\| \le \|X\|\|h\|$ for all $n \ge 0$. Since cT is quasi-hyponormal, $\|g_0\| \ge \|g_1\| \ge \|g_2\| \ge \cdots$ by Lemma 2.2, and the first inequality in this chain, $\|g_1\| \le \|g_0\|$, shows that $\|XS^{-1}h\| \le \|S^{-1}\|\|Xh\|$.

Proposition 2.4. *If S in $\mathcal{L}(H)$ is invertible, T in $\mathcal{L}(K)$ is a quasi-hyponormal, and X in $\mathcal{L}(H,K)$ has dense range and satisfies $XS = TX$, then T is invertible.*

Proof. Suppose that $\dim H \ge 1$. Since $X(H) = X(S(H)) = TX(H) \subseteq T(K)$, the range of T contains the range of X, and so T has dense range. It remains only to show that T is bounded below, and by continuity it will suffice to show that T is bounded below on the range of T. By Lemma 2.3,

$$\|XS^{-1}h\| \le \|S^{-1}\|\|Xh\| = \|S^{-1}\|\|XSS^{-1}h\| = \|S^{-1}\|\|TXS^{-1}h\|$$

for all h in H. Put $h_1 = S^{-1}h$. Then $\|S^{-1}\|^{-1}\|Xh_1\| \le \|TXh_1\|$, and so T is bounded below on the range of X.

If M is a closed subspace of H, $H = M \oplus M^{\perp}$ If T is in $\mathcal{L}(H)$, then T can be written as a 2×2 matrix with operator entries,

$$T = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$

where $W \in \mathcal{L}(M)$, $X \in \mathcal{L}(M^{\perp}, M)$, $Y \in \mathcal{L}(M, M^{\perp})$, and $Z \in \mathcal{L}(M^{\perp})$([2]).

Proposition 2.5. *If S and T are quasi-similar quasi-hyponormal operators in $\mathcal{L}(H)$ and* $\ker(S) = \ker(T)$, *then $S_1 = S|_{\ker(S)^\perp}$ and $T_1 = T|_{\ker(T)^\perp}$ are quasi-similar quasi-hyponormal operators.*

Proof. Since S and T are quasi-similar, there exists quasi-invertible operators X and Y such that $XS = TX$ and $SY = YT$. the $\ker(S)$ is invariant under both X and Y. Thus the matrices of S, T, X and Y with respect to decomposition $H = \ker(S)^\perp \oplus \ker(S)$ are

$$\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} X_1 & 0 \\ X_2 & X_3 \end{pmatrix}, \text{ and } \begin{pmatrix} Y_1 & 0 \\ Y_2 & Y_3 \end{pmatrix},$$

respectively. It is easy to verify that the ranges of X_1 and Y_1 are dence in $\ker(S)^\perp$. We now show that $\ker(X_1) = \ker(Y_1) = \{0\}$. Suppose that $z \in \ker(X_1)$. Then $TX(z \oplus 0) = 0$. The equation $XS = TX$ implies that $z \in \ker(S_1)$. This implies that $z = 0$, and so $\ker(X_1) = \{0\}$. Likewise $\ker(Y_1) = \{0\}$. Therefore X_1 and Y_1 are quasi-invertible operators on $\ker(S)^\perp$ and equations $XS = TX$ and $SY = TY$ imply that $X_1S_1 = T_1X_1$ and $S_1Y_1 = Y_1T_1$. Hence S_1 and T_1 are quasi-similar. By Theorem 2.1, the operators S_1 and T_1 are quasi-hyponormal.

Theorem 2.6. *Let H and K be Hilbert spaces. Suppose that S and T are quasi-similar quasi-hyponormal operators on Hilbert spaces in $\mathcal{L}(H)$. Then S is a Fredholm operator satisfying ind(S) $= 0$ if and only if T is a Fredholm operator satisfying ind(S) $= 0$.*

Proof. Since S and T are quasi-similar, there exist quasi-invertible operator X and Y such that $XS = TX$ and $SY = YT$. Now suppose that S is a Fredholm operator satisfying $\operatorname{ind}(S) = 0$. Since S and T are quasi-similar, it follows that $\dim(\ker(S)) = \dim(\ker(T))$ and $\dim(\ker(S)^\perp) = \dim(\ker(T)^\perp)$. Thus without loss of generality, we may assume that $\ker(S) = \ker(T)$. (Replace T by U^*TU, X by U^*X and Y by YU for a suitable unitary operator U if necessary). Since S is a quasi-hyponormal Fredholm operator with $\operatorname{ind}(S) = 0$, we have $\ker(S) = \ker(S^*)$. Note that $\ker(S)$ is an invariant subspace for the operator X. The matrices of S, T and X with respect to the decomposition $H = \ker(S)^\perp \oplus \ker(T)$ are

$$\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and } \begin{pmatrix} X_1 & 0 \\ X_2 & X_3 \end{pmatrix},$$

respectively, where S_1 is invertible, T_1 is quasi-hyponormal and X_1 has dense range in $\ker(S)^{\perp}$. The equation $XS = TX$ implies that $X_1S_1 = T_1X_1$. Hence, by Proposition 2.4, T_1 is also invertible. Thus T is a Fredholm operator satisfying $\mathrm{ind}(T) = 0$. Hence, by symmetry, the proof is complete.

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