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Computer Aided Mathematical Proof - in Finding the Inverse Matrix of a Special Matrix

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I. Introduction

As we are studying mathematics, we are used to meet some problems, say P(n) for whose proof mathematical induction is related. Whenever we need the general terms of, for examples, sequences, series or something of n by n matrices, their first sevral properties make conjecture on what the general terms are. Sometimes it is so tedious to find out first sevral properties that we may fail to get them exactly, and thus to be not able to obtain the general form correctly. However if we use a computer package, it makes us to be easy to get them. The computers have been highly developed so that we can compute whatever we want very fast. Moreover it is possible to compute symbolically. Also we are easy to get PC because of their low price relatively. We'll suggest thus in this work how to get some general forms of the problem P(n) as showing how to get, for example, the determinant and the inverse matrix of the following matrix M with an aid of computer. The matrix may be used to estimate coloured noise in a field of randomness.

$$M \equiv \begin{pmatrix} 1 & a & \cdots & a^{n-1} & a^n \\ a & 1 & \cdots & a^{n-2} & a^{n-1} \\ \vdots & & & & \vdots \\ a^{i-1} & \cdots & \cdots & \cdots & a^{n+1-i} \\ \vdots & & & & \vdots \\ a^{n-1} & a^{n-2} & \cdots & 1 & a \\ a^n & a^{n-1} & \cdots & a & 1 \end{pmatrix} = (a^{|i-j|})$$

II. Determinant and Inverse Matrix

It is very tedious to find out in hand the determinant or the inverse of the matrix M for n = 3, 4 even higher. However we are able to get easily first several determinants with an aid of symbolic computation[1] which are

n	Det (M)
0	1
1	$1 - a^2$
2	$1 - 2a^2 + a^4$
3	$1 - 3a^2 + 3a^4 - a^6$
4	$1 - 4a^2 + 6a^4 - 4a^6 + a^8$

The above results makes us consider the general form of the determinant of the matrix M as

$$\begin{aligned} \mathrm{Det}(M) &= \sum_{r=0}^{n} {}_nC_r(-1)^r a^{2r} \\ &= (1 - a^2)^n. \end{aligned}$$

We also get the inverse matrix multiplied by the determinant of the matrix M for the sake of simplicity as following table 1:

1) MATHEMATICA of Wolfram Research, Inc.

n	$\mathrm{Det}(M)\cdot M^{-1}$
1	$\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix}$
2	$\begin{bmatrix} 1-a^2 & -a+a^3 & 0 \\ -a+a^3 & 1-a^4 & -a+a^3 \\ 0 & -a+a^3 & 1-a^2 \end{bmatrix}$
3	$\begin{bmatrix} 1-2a^2+a^4 & -a+2a^3-a^5 & 0 & 0 \\ -a+2a^3-a^5 & 1-a^2-a^4+a^6 & -a+2a^3-a^5 & 0 \\ 0 & -a+2a^3-a^5 & 1-a^2-a^4+a^6 & -a+2a^3-a^5 \\ 0 & 0 & -a+2a^3-a^5 & 1-2a^2+a^4 \end{bmatrix}$

<table 1>

It follows from the results that the inverse matrix of M may have a structure of

$$M^{-1} = \frac{1}{\mathrm{Det}(M)} \begin{bmatrix} f & g & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ g & h & g & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & & & & \vdots \\ \vdots & & & g & h & g & & & \vdots \\ \vdots & & & & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & g & h & g \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & g & f \end{bmatrix}.$$

It remains to get the general forms of the entries f, g and h. We already have their first a couple of terms, from which we can expect their general structures as following table 2.

$$f = \sum_{r=0}^{n-1} {}_{n-1}C_r(-1)^r a^{2r} = (1-a^2)^{n-1}$$

$$g = -a\cdot f$$

$$h = (1+a^2)(1-a^2)^{n-1} = (1+a^2)f$$

III. Experimental Results and Proof

We obtain experimentally the determinant and the inverse matrix of the matrix $M=(a^{|i-j|})$ for $i,j=0,1,\cdots,n$ which are

1. $\mathrm{Det}(M) = (1-a^2)^n$

2. $$M^{-1} = \frac{1}{1-a^2}\cdot \begin{bmatrix} 1 & -a & 0 & \cdots & \cdots & \cdots & 0 \\ -a & 1+a^2 & -a & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & -a & 1+a^2 & -a & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -a & 1+a^2 & -a \\ 0 & \cdots & \cdots & \cdots & 0 & -a & 1 \end{bmatrix}$$

It remains to show that the obtained result is theoretically correct. But it is easy to prove that

$$M^{-1}\cdot M = \frac{1}{1-a^2}\cdot \begin{bmatrix} 1 & -a & & & & & \\ -a & 1+a^2 & -a & & & & \\ & \ddots & \ddots & \ddots & & 0 & \\ & & -a & 1+a^2 & -a & & \\ & 0 & & \ddots & \ddots & \ddots & \\ & & & & -a & 1+a^2 & -a \\ & & & & & -a & 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & a & a^2 & a^3 & \cdots & \cdots & a^n \\ a & 1 & a & a^2 & \cdots & \cdots & a^{n-1} \\ \vdots & & \ddots & \ddots & & & \vdots \\ a^{i-1} & \cdots & \cdots & 1 & a & \cdots & a^{n+1-i} \\ \vdots & & & & \ddots & \ddots & \vdots \\ a^{n-1} & a^{n-2} & \cdots & \cdots & \cdots & 1 & a \\ a^n & a^{n-1} & \cdots & \cdots & \cdots & a & 1 \end{bmatrix}$$

$$= \frac{1}{1-a^2}\cdot \mathrm{Diag}(1-a^2)$$

$$= I$$

n	f	g	h
1	1	$-a$	
2	$1-a^2$	$-a+a^3$	$(1+a^2)(1-a^2)^1$
3	$1-2a^2+a^4$	$-a+2a^3-a^5$	$(1+a^2)(1-a^2)^2$
4	$1-3a^2+3a^4-a^6$	$-a+3a^3-3a^5+a^7$	$(1+a^2)(1-a^2)^3$

<table 2>

Therefore we get the determinant and the inverse of the given matrix M.

Ⅳ. Conclusion Remarks

We suggest in the above how to get some results with an aid of symbolic computation. That is, the first is to take first sevral results experimentally using symbolic computation on computer. The second is to conjecture the general form from the obtained results. The other is to prove that the general form is theoretically correct.

The symbolic computation may be very useful in mathematical education. It reduces the tedious work especially in calculation so that it may help doing research well on some topics. Thus we suggest an education on a computer to be contained in the curriculum of mathematics education and recommend to use mathematical packages like Mathematica, Matlab or Maxyma, so on in classes.

Reference

Wolfram, S (1991). Mathemtica : A System for Doing Mathematics by Computer, 2nd ed., Addison Wesley.

Appendix (Commands of Mathematica)

- givmat[a_ ,n_]:=
  Table[a^Abs[i-j],{i,0,n},{j,0,n}]1
- matinv[a_ ,n_]:=
  Inverse[givmat[a,n]]1
- detmat[a_ ,n_]:= Det[givmat[a,n]]]1
- invmat[a_ ,n_]:= Table[Switch[j-i,
  -1,-a,
  0,If[i==0 ||i==n,1,1+a^2],
  1,-a,
  _, 0],
  {i,0,n},{j,0,n}]/(1-a^2)\end{verbatim}
- givmat[a,3].invmat[a,3]
- Simplify[%]