Journal of the Korean
Statistical Society
Vol. 24, No. 2, 1995

Consistency and Bounds on the Bias of S^2 in the Linear Regression Model with Moving Average Disturbances

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ABSTRACT

The ordinary least squares based estimate S^2 of the disturbance variance is considered in the linear regression model when the disturbances follow the first-order moving-average process. It is shown that S^2 is weakly consistent estimate for the disturbance variance without any restriction on the regressor matrix X. Also, simple exact bounds on the relative bias of S^2 are given in finite sample sizes.

KEYWORDS: Ordinary least square estimator, Linear regression, Disturbance variance, Moving average process, Bias of S^2, Consistency.

1. INTRODUCTION

We consider the linear regression model

$$y = X\beta + u, \tag{1.1}$$

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where y is the $T \times 1$ vector of observations on the dependent variable and X is a nonstochastic $T \times k$ regressor matrix with rank $k \leq T$. The $k \times 1$ vector β contains the unknown regression coefficients and u is a $T \times 1$ disturbance vector with $E(u) = 0$ and covariance matrix $E(uu') = \sigma_u^2 V$, where V is assumed to be symmetric and positive definite.

It is well known that the Ordinary Least Square (OLS)-based estimator of σ_u^2,

$$S^2 = \frac{1}{T-k}(\hat{u}'\hat{u}) = \frac{1}{T-k}(y - X\hat{\beta})'(y - X\hat{\beta}), \tag{1.2}$$

where $\hat{\beta} = (X'X)^{-1}X'y$, is a biased and inconsistent estimator of σ_u^2, when the disturbances have a non-scalar covariance matrix $\sigma_u^2 V$ (see Fomby et. al. (1984, p. 19) and Dhrymes (1978, p. 107)).

We investigate the relative bias of S^2 for σ_u^2. This bias leads to distortions of statistical inferences including the t-test and F-test for the parameters in the model (1.1). Several authors studied the expected value of S^2 when the disturbances are correlated. Cochran and Orcutt (1949), in an sampling experiment, found that serial correlation results in an underestimation of σ_u^2. Watson (1955) and Sathe and Vinod (1974) obtained bounds for $E(S^2)$ in terms of the characteristic roots of the covariance matrix of the disturbances. Neudecker (1977, 1978) also provided numerical evaluations of the upper and lower bounds on the relative bias $E(S^2/\sigma_u^2)$ for the case where the disturbances follow a first-order autoregressive processes. We present in this paper that the the exact upper and lower bounds for $E(S^2/\sigma_u^2)$ in finite sample sizes when the disturbances u are generated by an invertible first-order moving-average, $MA(1)$, process.

Secondly, more important issues lie in some asymptotic properties of the estimator S^2. Recently in the case of the $MA(1)$-disturbance, Song (1994) has shown the asymptotic unbiasedness of S^2. Further in this paper we will provide that S^2 is shown to be weakly consistent for σ_u^2, regardless of the regressor matrix X.

2. BOUNDS FOR THE BIAS OF S^2 IN FINITE SAMPLES

Let u in (1.1) be generated by an invertible $MA(1)$ process such as

$$u_t = \varepsilon_t - \theta\varepsilon_{t-1}, \qquad t = 1, 2, \cdots, T, \tag{2.1}$$

where $|\theta| < 1$ and $\{\varepsilon_t\}$ is a sequence of independent and identically distributed random variables with $E(\varepsilon_t) = 0, \mathrm{Var}(\varepsilon_t) = \sigma_\varepsilon^2$.

The $T \times T$ autocovariance matrix is given by

$$E(uu') = \sigma_u^2 V = \sigma_\varepsilon^2 W, \tag{2.2}$$

where σ_ε^2 is $\sigma_u^2/(1+\theta^2)$ and the form of W is

$$W = \begin{bmatrix} 1+\theta^2 & -\theta & 0 & 0 & \cdots & 0 & 0 \\ -\theta & 1+\theta^2 & -\theta & 0 & \cdots & 0 & 0 \\ 0 & -\theta & 1+\theta^2 & -\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1+\theta^2 & -\theta \\ 0 & 0 & 0 & 0 & \cdots & -\theta & 1+\theta^2 \end{bmatrix}. \tag{2.3}$$

From the results of Watson (1955), Sathe and Vinod (1974) and Neudecker (1977, 1978) we have an interval for the expected value of the OLS-estimator over σ_u^2:

$$\frac{1}{T-k}\sum_{i=1}^{T-k}\lambda_{i+k} \le E\Big(\frac{S^2}{\sigma_u^2}\Big) = \frac{1}{T-k}\mathrm{tr}(MV) \le \frac{1}{T-k}\sum_{i=1}^{T-k}\lambda_i, \tag{2.4}$$

where $M = I - X(X'X)^{-1}X'$ and $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_T$ are ordered characteristic roots of V. In interval (2.4) can be shown that the lower bound is means of $T-k$ least of characteristic roots and upper bound is means of $T-k$ greatest of characteristic roots. This bounds show that the bias can be positive and negative, depending on the regressor matrix X and V. When extra information on V is avarible the relative bias $E(S^2/\sigma_u^2)$ can be tabulated.

To derive the exact upper and lower bounds for $E(S^2/\sigma_u^2)$ we need the characteristic roots of V in (2.2) which can be obtained from Song (1994) as

$$\lambda_i(V) = \lambda_i\Big(\frac{1}{1+\theta^2}W\Big) = \frac{1+\theta^2+2|\theta|\cos(\frac{i\pi}{T+1})}{1+\theta^2}, \quad i = 1,2,\ldots,T, \qquad (2.5)$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_T$ are ordered characteristic roots of V.

2.1 Bias of S^2 in Homogeneous Regression Model

First we have computed bounds for $E(S^2/\sigma_u^2)$ according to (2.4) and (2.5) when no intercept in the regression (i.e. homogeneous regression). Table I gives the numerical values of bounds for selected values of T, k and θ. Some of results (for $k=2$ and $k=4$) are represented in Figure 1 and 2. Moreover, one can easily see by the figures that the intervals are symmetrical around one. This show that the interval becomes wider for increasing values of $|\theta|$ and k, given T. On the otherhand, the interval is getting narrow as the sample size increases for given θ and k.

2.2 Bias of S^2 in Inhomogeneous Regression Model

Moreover we want to incorporate a constant term in the regression, which implies that X should contain a column of ones (i.e. inhomogeneous regression). We shall derive from (2.4) an interval for this case. It is easy to see that $\mathrm{tr}(MV) = \mathrm{tr}(MAVA)$, where $A = I - (1/T)ee'$ and e is a column of ones. Then we can translate the inequalities (2.4) into (see Neudecker, 1978)

$$\frac{1}{T-k}\sum_{i=1}^{T-k}\delta_{i+k-1} \leq E\Big(\frac{S^2}{\sigma_u^2}\Big) = \frac{1}{T-k}\mathrm{tr}(MAVA) \leq \frac{1}{T-k}\sum_{i=1}^{T-k}\delta_i, \qquad (2.6)$$

where $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{T-1} \geq \delta_T = 0$ are ordered characteristic roots of AVA. To the zero characteristic roots corresponds the eigenvector e. In interval (2.6) is tighter than the inteval (2.4), because the constraint $y'e = 0$ have been imposed.

We have computed bounds for $E(S^2/\sigma_u^2)$ according to (2.5) and (2.6) for various values of T, k and θ. The results are shown in Table II and III. Some of results (for $k = 2$ and $k = 4$) are represented in Figure 3, 4, 5 and 6. In case of $\theta < 0$ (in Figure 3 and 4) can be shown that the intervals are symmetrical around one. In case of $\theta > 0$ (in Figure 5 and 6) we can show that the intervals are asymmetrical around one. They clearly suggest a negative bias. In this case, also for given T the bounds for the relative bias become wider as the number of parameter k and θ increases. In contrary to, this bounds for given θ and k become narrower as T increases.

Table I. The Bounds of $E(\frac{S^2}{\sigma_u^2})$ for Various T, k and θ in Homogeneous Regression Model.

		$\theta = 0.3(-0.3)$		$\theta = 0.5(-0.5)$		$\theta = 0.8(-0.8)$	
	k	lower	upper	lower	upper	lower	upper
T=10	2	0.87610	1.12390	0.81993	1.18007	0.78040	1.21960
	3	0.80690	1.19310	0.71936	1.28064	0.65775	1.34224
	4	0.73660	1.26340	0.61720	1.38280	0.53317	1.46683
	5	0.66826	1.33174	0.51787	1.48213	0.41203	1.58797
T=20	2	0.91935	1.07963	0.88279	1.11528	0.85706	1.14131
	3	0.90787	1.09213	0.86610	1.13390	0.83671	1.16329
	4	0.87368	1.12632	0.81642	1.18358	0.77612	1.22388
	5	0.83836	1.16164	0.76508	1.23492	0.71352	1.28648
T=30	2	0.95300	1.04700	0.93170	1.06830	0.91671	1.08329
	3	0.94029	1.05971	0.91323	1.08677	0.89418	1.10582
	4	0.91854	1.08146	0.88162	1.11838	0.85563	1.14437
	5	0.89603	1.10397	0.84890	1.15110	0.81573	1.18427
T=70	2	0.97717	1.02283	0.96682	1.03318	0.95954	1.04046
	3	0.97546	1.02453	0.96434	1.03566	0.95652	1.04348
	4	0.96688	1.03312	0.95187	1.04813	0.94131	1.05869
	5	0.95811	1.04189	0.93912	1.06088	0.92576	1.07424

Table II. The Bounds of $E(\frac{S^2}{\sigma_u^2})$ for Various T, k and $\theta < 0$ in Inhomogeneous Regression Model.

		$\theta = -0.3$		$\theta = -0.5$		$\theta = -0.8$	
	k	lower	upper	lower	upper	lower	upper
T=10	2	0.81320	1.11980	0.78518	1.17412	0.76547	1.21229
	3	0.73501	1.19061	0.67965	1.27702	0.64067	1.33778
	4	0.65364	1.26049	0.57218	1.37858	0.51485	1.46162
	5	0.56870	1.33036	0.46385	1.48013	0.39066	1.58545
T=20	2	0.88194	1.07863	0.86330	1.11428	0.85113	1.14031
	3	0.87980	1.09115	0.83812	1.13248	0.83241	1.16156
	4	0.84389	1.12527	0.78674	1.18207	0.77160	1.22203
	5	0.80658	1.16082	0.76586	1.23373	0.70870	1.28503
T=30	2	0.93211	1.04621	0.92160	1.06716	0.91394	1.08190
	3	0.92296	1.05921	0.90483	1.08605	0.89207	1.10494
	4	0.90055	1.08094	0.87290	1.11764	0.85344	1.14346
	5	0.87732	1.10352	0.83984	1.15045	0.81346	1.18347
T=70	2	0.96756	1.02260	0.96232	1.03287	0.95862	1.04008
	3	0.97692	1.02424	0.96025	1.03485	0.95525	1.04205
	4	0.95687	1.03289	0.94760	1.04733	0.94066	1.05800
	5	0.94834	1.04132	0.93561	1.05926	0.92444	1.07401

Table III. The Bounds of $E(\frac{S^2}{\sigma_u^2})$ for Various T, k and $\theta > 0$ in Inhomogeneous Regression Model.

		$\theta = 0.3$		$\theta = 0.5$		$\theta = 0.8$	
	k	lower	upper	lower	upper	lower	upper
T=10	2	0.66233	1.00409	0.63262	1.00594	0.57938	1.00725
	3	0.63385	1.07062	0.53263	1.10263	0.46140	1.12517
	4	0.55801	1.14246	0.43321	1.20705	0.34538	1.25251
	5	0.48529	1.21562	0.34262	1.31336	0.24222	1.38215
T=20	2	0.88139	1.00200	0.75672	1.00290	0.73538	1.00354
	3	0.82125	1.03219	0.76980	1.04678	0.73218	1.05701
	4	0.78672	1.06520	0.71836	1.09475	0.60773	1.11554
	5	0.74962	1.09983	0.66634	1.14509	0.60773	1.17694
T=30	2	0.88778	1.00078	0.85663	1.00113	0.83470	1.00138
	3	0.88501	1.02054	0.84971	1.02989	0.82480	1.03642
	4	0.86199	1.04153	0.81707	1.09221	0.78511	1.07361
	5	0.83840	1.06342	0.78327	1.06040	0.74490	1.11241
T=70	2	0.94525	1.00020	0.92989	1.00030	0.91906	1.00036
	3	0.96546	1.00536	0.94590	1.00141	0.93440	1.01560
	4	0.95182	1.01619	0.93480	1.02324	0.91721	1.03101
	5	0.94280	1.023191	0.92463	1.03142	0.90217	1.04284

Figure 1. The Bounds of $E(\frac{S^2}{\sigma_u^2})$ in Homogeneous Regression Model for $k = 2$.

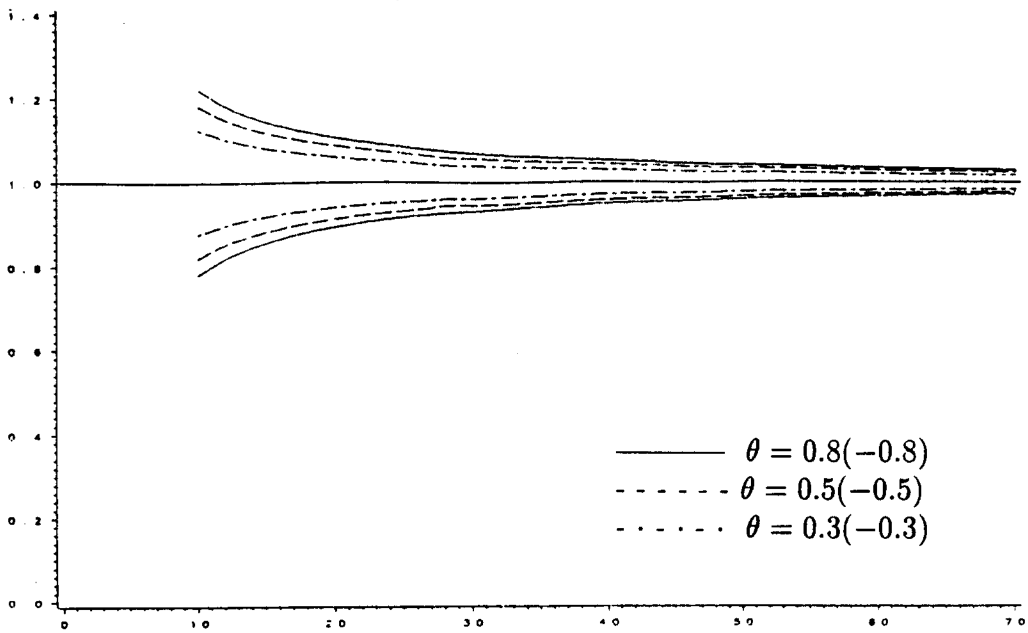


Figure 2. The Bounds of $E(\frac{S^2}{\sigma_u^2})$ in Homogeneous Regression Model for $k = 4$.

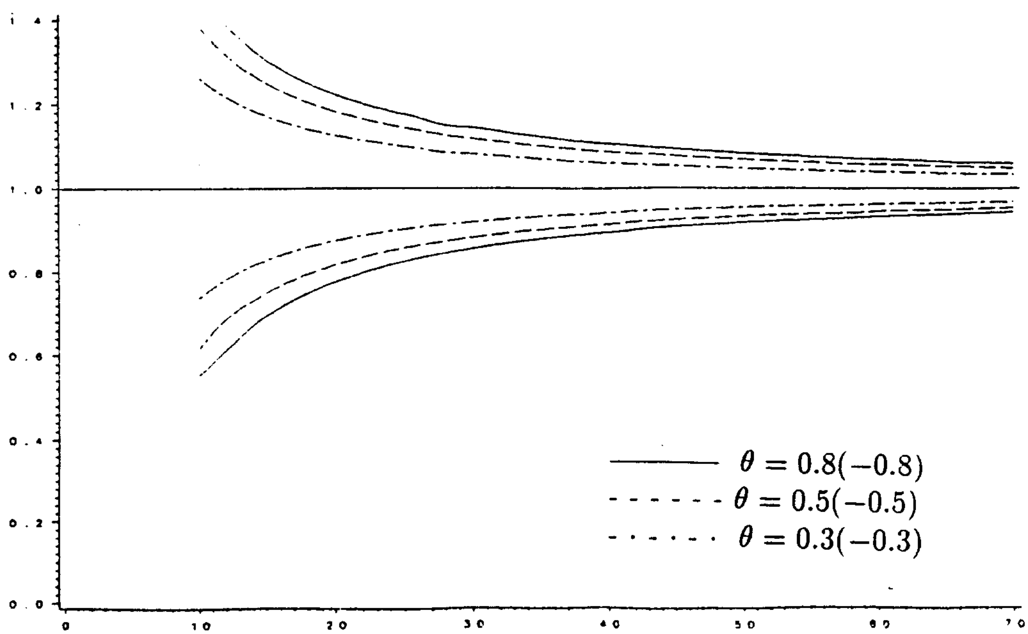


Figure 3. The Bounds of $E(\frac{S^2}{\sigma_u^2})$ in Inhomogeneous Regression Model for $k = 2$ and $\theta < 0$.

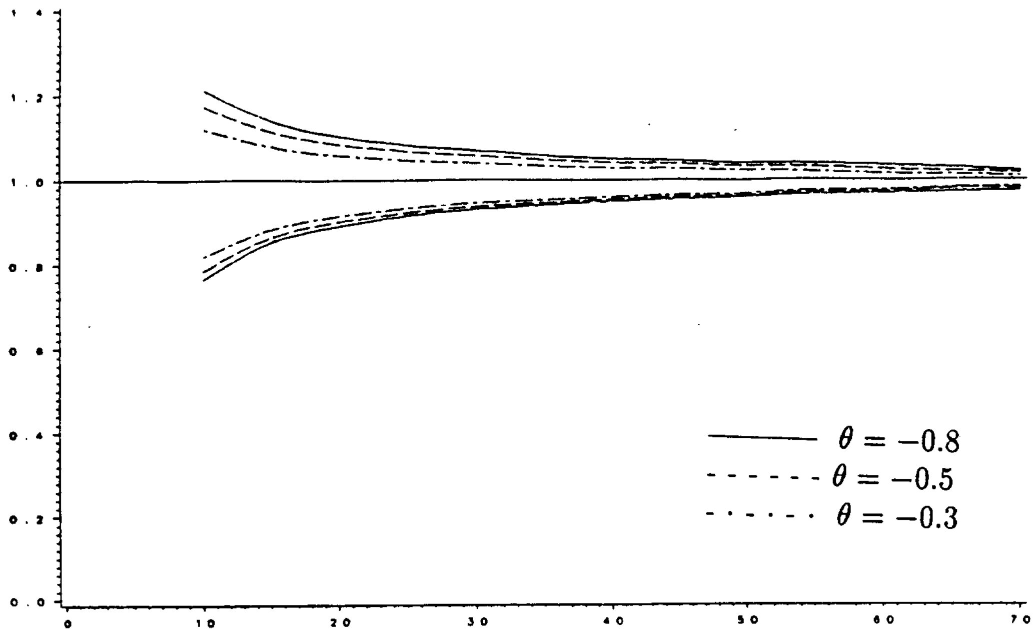


Figure 4. The Bounds of $E(\frac{S^2}{\sigma_u^2})$ in Inhomogeneous Regression Model for $k = 4$ and $\theta < 0$.

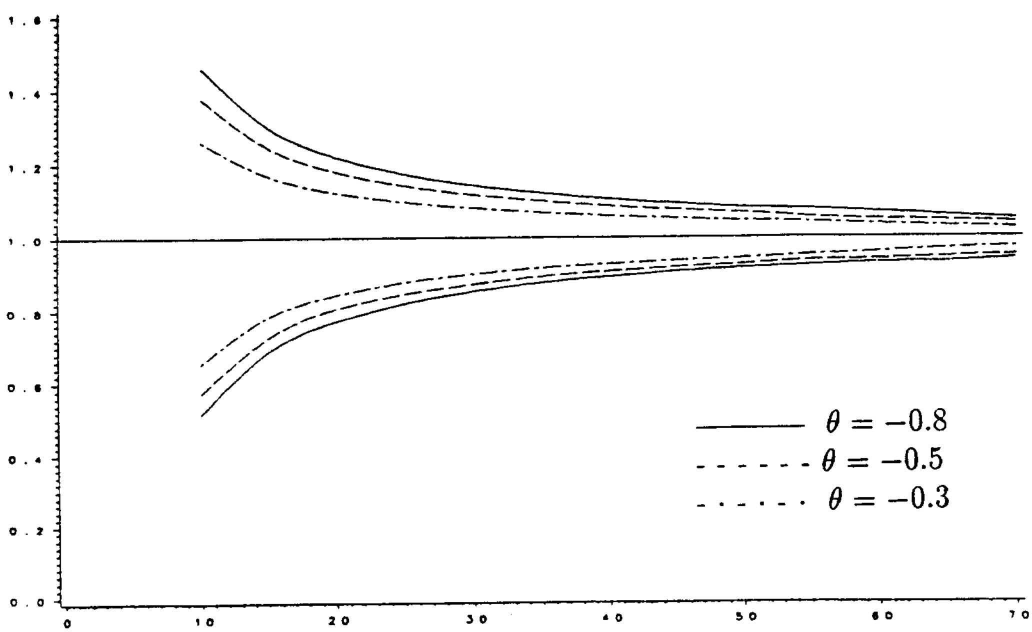


Figure 5. The Bounds of $E(\frac{S^2}{\sigma_u^2})$ in Inhomogeneous Regression Model for $k = 2$ and $\theta > 0$.

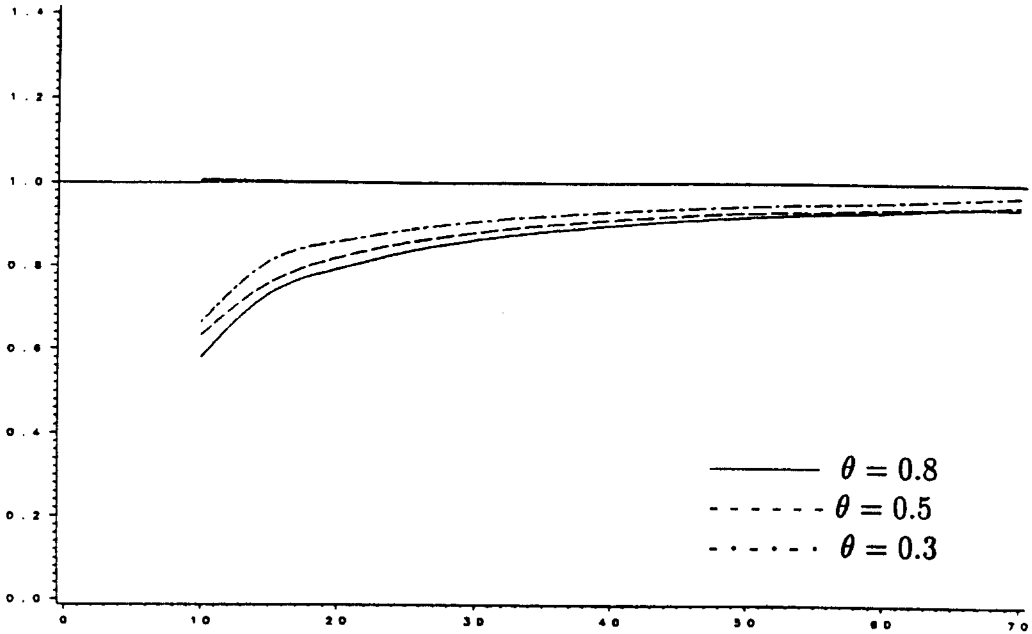
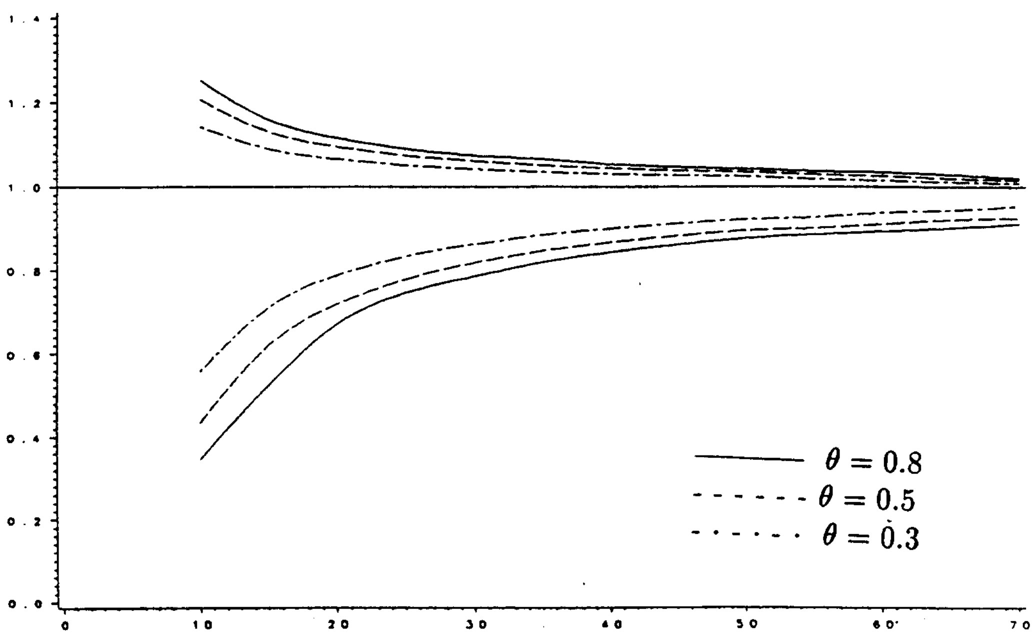


Figure 6. The Bounds of $E(\frac{S^2}{\sigma_u^2})$ in Inhomogeneous Regression Model for $k = 4$ and $\theta > 0$.



3. CONSISTENCY OF S^2

It has long been known (see e.g. Klock (1972) or Drygas (1975)) that S^2 is consistent for the true disturbance variance σ_u^2 under conditions which are much less restrictive than these needed for the consistency of $\widehat{\beta}$, (given serially uncorrelated disturbances). However, when the disturbances are correlated but still homoscedastic, with general covariance $E(uu') = \sigma_u^2 V$, S^2 often remains consistent depending on the structure of V (see Krämer and Berghoff, 1991). In what follows, we explore the consistency of S^2 in the context of the first order moving average disturbances. Then we have following theorem:

Theorem 3.1. If the disturbances u in model (1.1) follow an invertible $MA(1)$ process and $E(\varepsilon_t^4) = \eta < \infty$, then S^2 is a weakly consistent estimator for σ_u^2.

Proof. Using the results of Krämer and Berghoff (1991), one can easily show that S^2 is weakly consistent irrespective of X as follow.

First we must show that $\frac{u'u}{T} \xrightarrow{p} \sigma_u^2$.

$$\text{i)} \quad E\Big[\frac{1}{T}u'u\Big] = \frac{1}{T}E\Big[\sum_{t=1}^{T} u_t^2\Big] = \frac{1}{T}E\Big[\sum_{t=1}^{T}(\varepsilon_t - \theta\varepsilon_{t-1})^2\Big] (\text{since } E(\varepsilon_t\varepsilon_{t-1}) = 0)$$

$$= \frac{1}{T}E\Big[\sum_{t=1}^{T}(\varepsilon_t^2 + \theta^2\varepsilon_{t-1}^2)\Big] = (1+\theta^2)\sigma_\varepsilon^2 = \sigma_u^2.$$

$$\text{ii)} \quad \mathrm{Var}\Big[\frac{1}{T}u'u\Big] = E\Big[\Big(\frac{1}{T}u'u\Big)^2\Big] - \Big[E\Big(\frac{1}{T}u'u\Big)\Big]^2 = E\Big[\Big(\frac{1}{T}u'u\Big)^2\Big] - \sigma_u^4.$$

It remains to show that $E\Big[\Big(\frac{1}{T}u'u\Big)^2\Big] \xrightarrow{p} \sigma_u^4$.

$$E\Big[\Big(\frac{1}{T}u'u\Big)^2\Big] = \frac{1}{T^2}\sum_{t=1}^{T} E\Big[(\varepsilon_t - \theta\varepsilon_{t-1})^4\Big] + \frac{2}{T^2}\sum_{t<}\sum_{s}\Big[(\varepsilon_t - \theta\varepsilon_{t-1})^2(\varepsilon_s - \theta\varepsilon_{s-1})^2\Big].$$

$$\text{i)} \quad \frac{1}{T^2}\sum_{t=1}^{T} E\Big[(\varepsilon_t - \theta\varepsilon_{t-1})^4\Big] (\text{since } E(\varepsilon_t\varepsilon_{t-1}) = 0)$$

$$= \frac{1}{T^2}\sum_{t=1}^{T} E\left[\varepsilon_t^4 + 6\varepsilon_t^2\theta^2\varepsilon_{t-1}^2 + \theta^4\varepsilon_{t-1}^4\right] (\text{since } E(\varepsilon_t^4) = \eta < \infty)$$
$$= \frac{1}{T}\left[\eta(1+\theta^4) + 6\sigma_\varepsilon^4(\theta^2)\right] \xrightarrow{p} 0 \text{ for } T \to \infty.$$

ii) $$\frac{2}{T^2}\sum_{t<}\sum_{s} E\left[(\varepsilon_t - \theta\varepsilon_{t-1})^2(\varepsilon_s - \theta\varepsilon_{s-1})^2\right]$$
$$= \frac{2}{T^2}\sum_{t<}\sum_{s} E\left[\varepsilon_t^2\varepsilon_s^2 + (\theta\varepsilon_{s-1})^2\varepsilon_t^2 + (\theta\varepsilon_{t-1})^2\varepsilon_s^2 + (\theta\varepsilon_{t-1})^2(\theta\varepsilon_{s-1})^2\right]$$
$$= \frac{T(T-1)}{T^2}\left[\sigma_\varepsilon^4(1+\theta^4+2\theta^2)\right] \xrightarrow{p} \sigma_\varepsilon^4(1+\theta^2)^2 = \sigma_u^4 \text{ for } T \to \infty.$$

Secondly, the maximum characteristic roots of V in (2.2) can be expressed as follows (see Song, 1994):

$$\lambda_{\max}(V) \le \frac{(1+|\theta|)^2}{1+\theta^2}, \tag{3.1}$$

so that, the maximum characteristic root of V is $o(T)$ (i.e. $\lambda_{\max}(V)/T \to 0$ for $T \to \infty$). This completes the proof.

4. CONCLUSION

In this paper is shown that, S^2 is weakly consistent for σ_u^2, without any restriction of the regressor matrix X, when the disturbances u are generated by an invertible $MA(1)$ process. In addition, the exact upper and lower bounds on $E(S^2/\sigma_u^2)$ are given, which depend only on the dimension of T, the number of regressor k and θ.

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