Journal of the Korean
Statistical Society
Vol. 24, No. 2, 1995

Three Remarks on Pitman Domination

Seongmo Yoo[1] and Herbert T. David[2]

ABSTRACT

Three remarks are offered, pertaining to classes of estimators Pitman-dominating a given estimator. The first remark concerns incorporating general loss in the construction of such classes. The second remark concerns Pitman domination comparisons amongst the members of such classes. The third remark concerns construction of such a class in the location-scale case.

KEYWORDS: Location-scale family, Loss function, Median-unbiased estimators, Pitman domination, Shrinkage estimators.

1. INTRODUCTION

According to Pitman (1937), an estimator X is closer than an estimator Y to a scalar parameter θ (or, in terminology used below, X Pitman-dominates Y) if

$$\Pr_\theta(|X-\theta| < |Y-\theta|) > 1/2, \quad \forall\theta.$$

[1]Department of Applied Statistics, Korea University, Jochiwon, 339–700, Korea.
[2]Department of Statistics, Iowa State University, Ames, IA 50011, U.S.A.

This criterion is now called the Pitman Closeness Criterion (PCC). It is generalized (see Rao *et al.* (1986) and Sen *et al.* (1989)) with respect to the loss function $L(\cdot,\cdot)$ as follows : Let X and Y, with joint density depending on a parameter vector $\theta \in \Theta$, be estimators of θ. X is closer than Y to θ with respect to the loss function $L(\cdot,\cdot)$ if

$$\Pr_\theta(L(X,\theta) < L(Y,\theta)) > 1/2, \quad \forall\theta \in \Theta.$$

This criterion is now called the Generalized Pitman Closeness Criterion (GPCC).

Recent interest in Pitman domination (David and Salem, 1991; Keating *et al.*, 1993; Robert *et al.*, 1993; Yoo and David, 1994) leads us to offer three remarks on the subject.

Generally, we address the construction of classes of estimators dominating a given estimator. Such constructions have been given, for example, in David and Salem (1991), Robert *et al.* (1993), Yoo and David (1994). David and Salem (1991) and Yoo and David (1994) suggest increasing the size of the class of Pitman-dominating estimators, by replacing a certain "boundary function" $\lambda^{-1}(\cdot)$ (corresponding to a certain set A in Robert *et al.* (1993)) by another boundary function $\mu(\cdot)$. Robert *et al.* (1993) and Yoo and David (1994) relax the condition that the dominating estimator be monotone in the given estimator. Also, Yoo and David (1994) examines the scale case and the non-central t case in addition to the location case.

The first remark pertains to applying the above ideas to the Rao *et al.* (1986) generalization of the Pitman closeness criterion to the case of general loss of the form $L(x,\theta) = h(x-\theta)$ where $h(y) = r(y)$ on $[0,+\infty)$ and $s(y)$ on $(-\infty,0]$, with $r(y)$ continuous increasing and $s(y)$ continuous decreasing, and $r(0) = s(0) = 0$. One finds that the boundary function $\mu(\cdot)$ is replaced by

$$\mu_L(x) = \min(c, \lambda^{-1}(x) + r^{-1}(s(x-\lambda^{-1}(x)))), \quad x \le c,$$

$$\mu_L(x) = \max(c, \lambda^{-1}(x) + s^{-1}(r(x-\lambda^{-1}(x)))), \quad x \ge c,$$

where c is arbitrary.

Let $\mathcal{S}$ be the set of estimators $U(X)$ with $(x, U(x))$ in A (A as in Robert *et al.* (1993)). The second remark explores three aspects of the comparison of members of $\mathcal{S}$. Let $U_2(X)$ be a member of $\mathcal{S}$ "near" $\lambda^{-1}(X)$, and let $U_1(X)$ be a member of $\mathcal{S}$ "near" X. Where "near" means "close to" in the sense of Euclidean distance. First, it may be claimed that, in a certain sense, an estimator $U_2(X)$ dominates more members of $\mathcal{S}$ than does an estimator $U_1(X)$. Second, it may be noted that the distribution of the random variable

$$Y_{U_2,U_1,\theta}(X) = |U_2(X) - \theta| - |U_1(X) - \theta|$$

tends to be skewed to the left, with smaller tails and more symmetry for large $|\theta|$. However, $U_1(X)$ is superior to $U_2(X)$ in the sense of Pitman nearness to X, as defined by Rao *et al.* (1986).

The third remark concerns estimating the location parameter for the location-scale case, in terms of the Sen *et al.* (1989) nuisance parameter formulation. It is shown that it is possible to dominate the natural estimator of the location parameter when the scale parameter is bounded below. It is further suggested by example that it may not be possible to do so when the lower bound is removed.

The above three sets of ideas can be exploited in conjunction with one another.

2. INCORPORATING LOSS

Propositions 1 and 2 in Yoo and David (1994) respectively concern constructing classes of estimators dominating (in the sense of Pitman (1937)) given estimators with distributions supported, respectively, by the real line and the positive half line. Their argument is easily extended to the construction of classes of shrinkage estimators dominating a given estimator in the sense of the generalized Pitman closeness criterion (Rao *et al.* (1986)) incorporating loss.

To that end, consider a loss function of the form $L(x,\theta) = h(x-\theta)$ where $h(y) = r(y)$ on $[0,+\infty)$ and $s(y)$ on $(-\infty,0]$, with $r(y)$ continuous increasing and $s(y)$ continuous decreasing, and $r(0) = s(0) = 0$.

Essentially as in Yoo and David (1994), let a_1 and a_2 satisfy $-\infty \leq a_1 < a_2 \leq +\infty$, and let I be the interval (a_1, a_2). Let X have density $f(x;\theta)$ supported by I, for all $\theta \in I$. Suppose further that there exists a real number c in I, and a continuous increasing function $\lambda(\theta)$ on I, such that

$$\int_{\lambda(\theta)}^{c} f(x;\theta)\,dx \leq 1/2, \quad a_1 < \theta \leq c, \tag{2.1a}$$

$$\int_{c}^{\lambda(\theta)} f(x;\theta)\,dx \leq 1/2, \quad c \leq \theta < a_2, \tag{2.1b}$$

$$\lambda(\theta) < \theta, \quad a_1 < \theta < c, \tag{2.1c}$$

$$\lambda(c) = c, \tag{2.1d}$$

$$\lambda(\theta) > \theta, \quad c < \theta < a_2. \tag{2.1e}$$

Now define $\mu_L(x)$ by

$$\mu_L(x) = \min(c, \lambda^{-1}(x) + r^{-1}(s(x - \lambda^{-1}(x)))), \quad a_1 < x \leq c, \tag{2.2a}$$

$$\mu_L(x) = \max(c, \lambda^{-1}(x) + s^{-1}(r(x - \lambda^{-1}(x)))), \quad c \leq x < a_2. \tag{2.2b}$$

Note that $\mu_L(x)$ is equal to $\mu(x)$ in Yoo and David (1994) when $h(\cdot)$ is symmetric about y-axis.

Proposition 1. Let X have density $f(x;\theta)$ as above. Then any estimator $T(X)$ of θ with $T(\cdot)$ continuous and

$$x < T(x) < \mu_L(x), \quad a_1 < x < c \tag{2.3a}$$

$$T(c) = c, \tag{2.3b}$$

$$\mu_L(x) < T(x) < x, \quad c < x < a_2, \tag{2.3c}$$

dominates the estimator X in the generalized Pitman sense; in other words, for $\theta \in (a_1, a_2)$,

$$\Pr_\theta(L(T(X), \theta) < L(X, \theta)) > 1/2. \tag{2.4}$$

Proof. When $\theta = c$, in view of (2.3) and the properties of the loss function $L(\cdot, \cdot)$, it is clear that

$$\Pr_\theta(L(T(X), \theta) < L(X, \theta)) = \int_{a_1}^{c} f(x; \theta)\, dx + \int_{c}^{a_2} f(x; \theta)\, dx = 1$$

and thus (2.4) holds.

As to $\theta \neq c$, the argument for $a_1 < \theta < c$ is essentially identical to that for $c < \theta < a_2$, and we choose the latter.

For $\theta \in (c, a_2)$, in view of (2.1b) and (2.4), it suffices to show that, given any $L(\cdot, \cdot)$,

$$\int_{a_1}^{c} f(x; \theta)\, dx + \int_{\lambda(\theta)}^{a_2} f(x; \theta)\, dx \leq \int_B f(x; \theta)\, dx \tag{2.5}$$

and

$$\int_B f(x; \theta)\, dx < \int_A f(x; \theta)\, dx, \tag{2.6}$$

where $A = \{x : L(T(x), \theta) < L(x, \theta)\}$ and $B = \{x : L(\mu_L(x), \theta) \leq L(x, \theta)\}$.

With regard to (2.5), we observe that, in view of (2.2b) and (2.3c),

$$L(\mu_L(X), \theta) \leq L(x, \theta)$$

at $x = \lambda(\theta)$. Also, for given θ and for $x > \lambda(\theta)$, we observe that $L(x, \theta)$ is increasing in x and $L(\mu_L(x), \theta)$ is smaller than $L(x, \theta)$ since $\lambda^{-1}(\cdot)$ is increasing, and by (2.2b). Additionally, in view of (2.3a), $L(\mu_L(x), \theta) < L(x, \theta)$ on $x \in (a_1, c)$. Thus, it is clear that both

$$a_1 < x < c$$

and

$$\lambda(\theta) \leq x < a_2$$

imply

$$L(\mu_L(x), \theta) \leq L(x, \theta).$$

Hence, (2.5) holds.

With regard to (2.6), it is not difficult to see, in view of (2.3c) and the property of the loss function $L(\cdot,\cdot)$, that B is a proper subset of A, which implies (2.6).

Example 1. Let X be a random variable having the normal density with unknown mean θ and unit variance. Let $L(x,\theta) = h(x-\theta)$ be the loss function of the form $h(y) = y^2$ on $[0, +\infty)$ and $h(y) = -y$ on $(-\infty, 0]$. It is noted that X is a median-unbiased estimator of θ. It is further noted that $a_1 = -\infty$ and $a_2 = +\infty$ in Proposition 1. Since X is a median-unbiased estimator of θ, it is possible to find a continuous increasing function $\lambda(x)$ satisfying (2.1). An example of the function $\lambda(x)$ is displayed in Figure 1. Now it is easy to note that

$$\mu_L(x) = \min(0, \lambda^{-1}(x) + \sqrt{\lambda^{-1}(x) - x}), \quad x \leq 0,$$

$$\mu_L(x) = \max(0, \lambda^{-1}(x) - (x - \lambda^{-1}(x))^2), \quad x \geq 0.$$

Then any continuous $T(X)$ satisfying (2.3) dominates X in the sense of GPCC. The boundary function $\mu_L(x)$ is displayed in Figure 2.

Example 2. Let $Y_1, \ldots, Y_n$ be iid random variables having the normal density with unknown mean θ and unknown variance σ^2. Let S^2 be the mean-unbiased estimator of σ^2 (i.e., $S^2 = \sum_{i=1}^{n}(Y_i - \bar{Y})^2/(n-1)$). It is noted by Pitman (1937) that

$$\Pr_\sigma(0 < S^2 < \sigma^2) = \Pr_\sigma\Big(0 < \frac{(n-1)S^2}{\sigma^2} < n-1\Big) > \frac{1}{2}.$$

Thus, there exist $k_n, 0 < k_n < 1$, such that

$$\Pr_\sigma\Big(0 < \frac{S^2}{k_n} < \sigma^2\Big) = \frac{1}{2}.$$

Figure 1. The Continuous Increasing Function $\lambda(x)$ in Example 1.

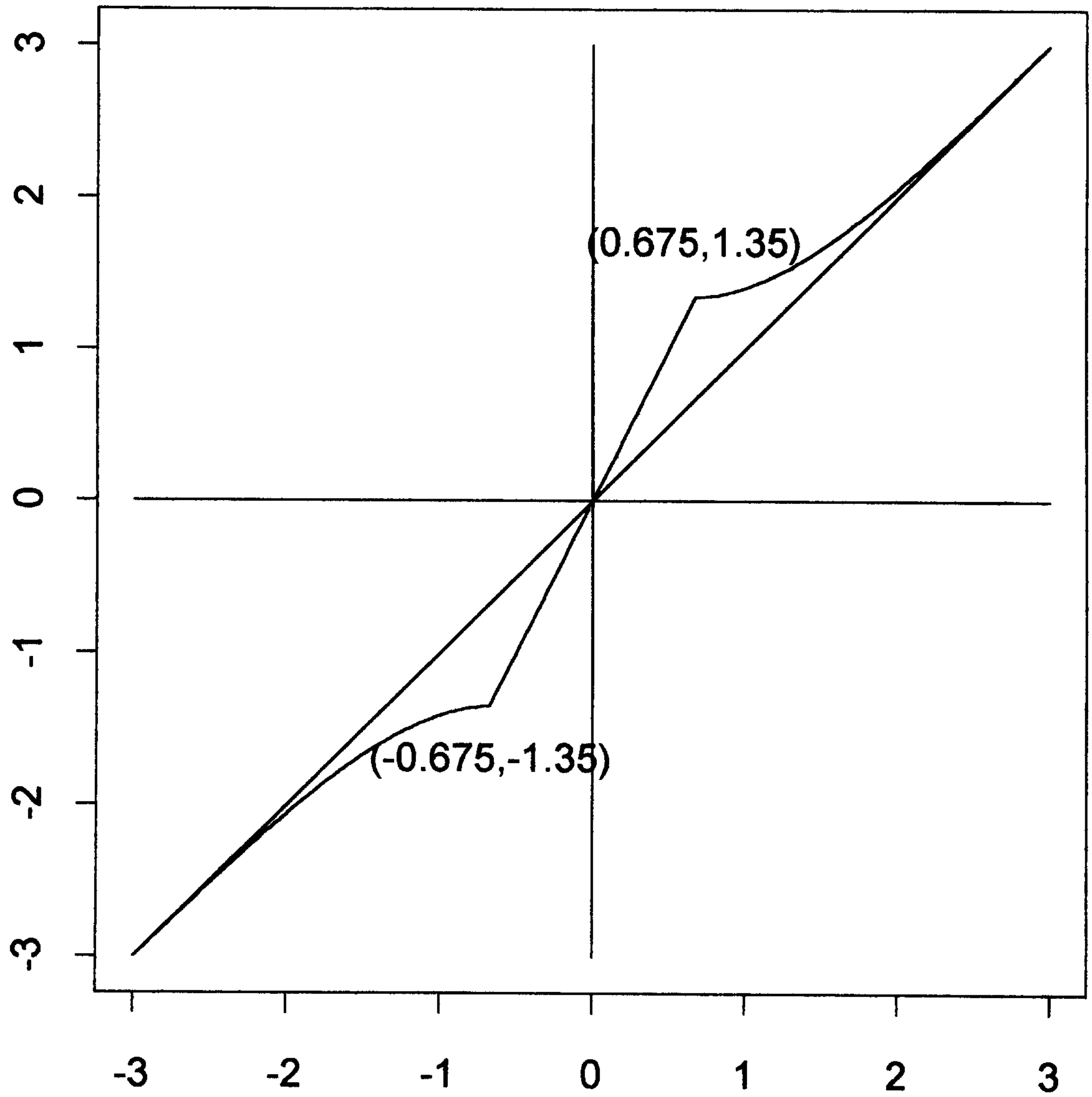


Figure 2. The Boundary Function $\mu_L(x)$ in Example 1.

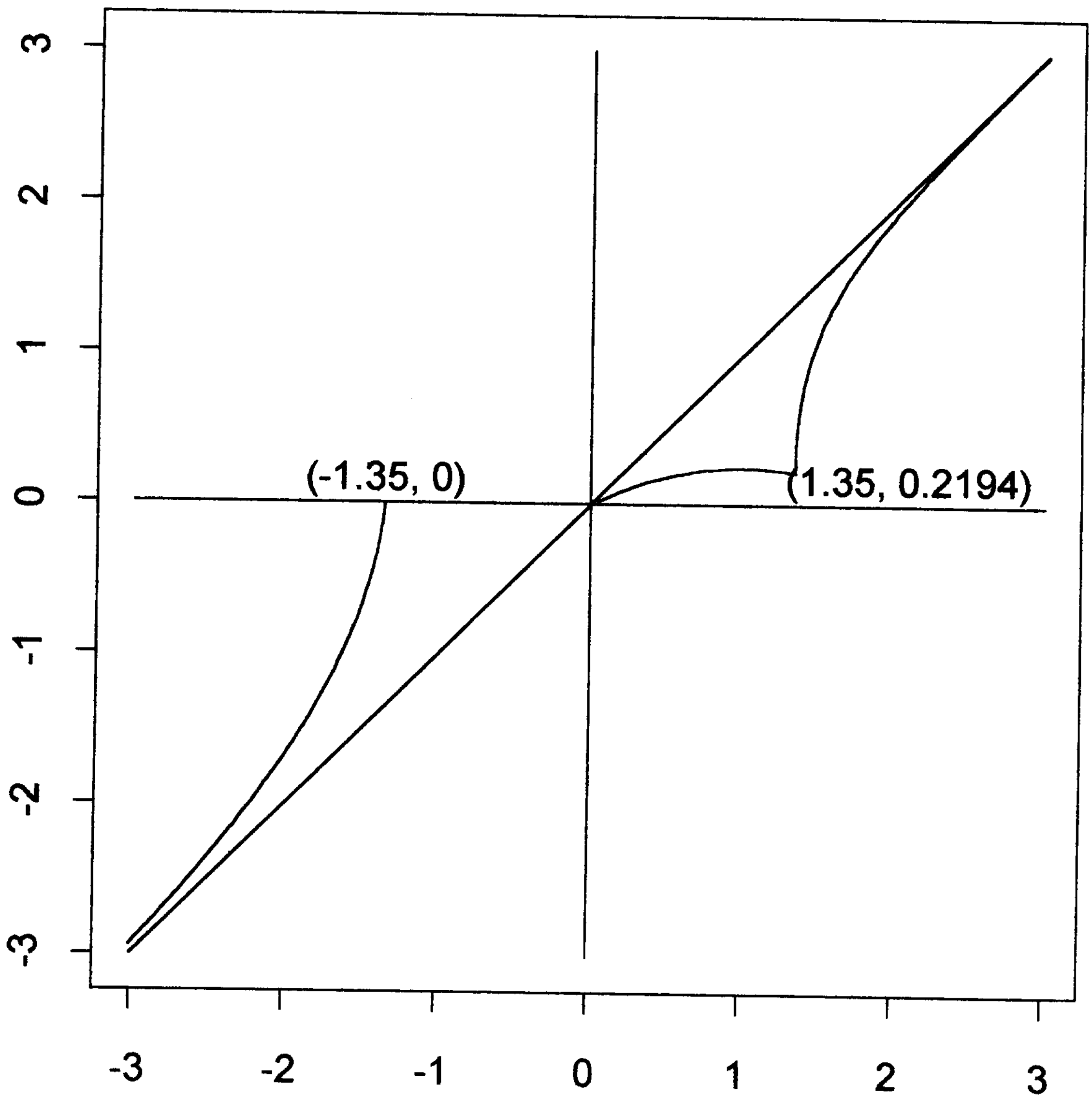
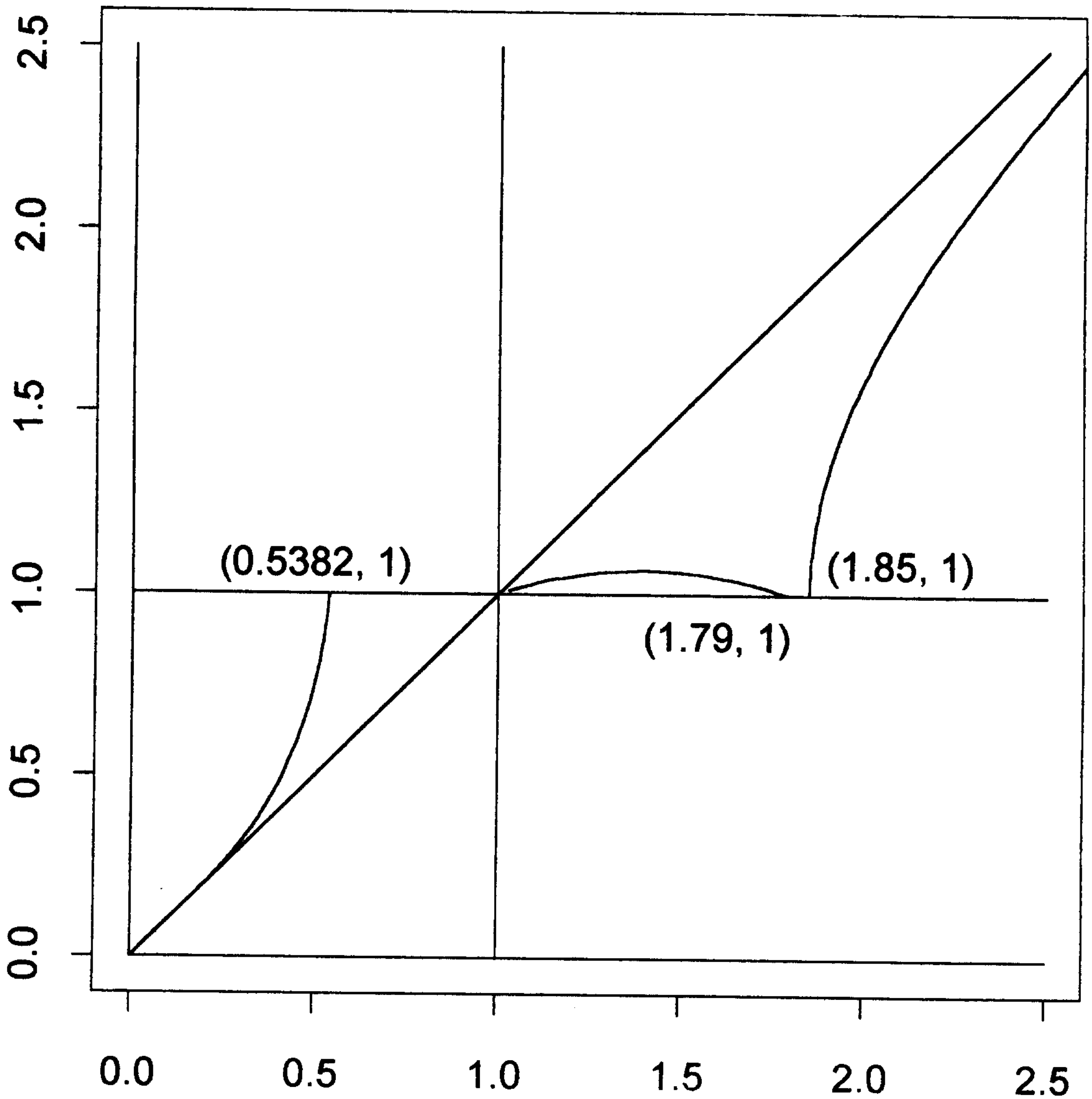


Figure 3. The Boundary Function $\mu_L(x)$ in Example 2.



Therefore, $\frac{S^2}{k_n}$ is a median-unbiased estimator of σ^2. Now it is clear that, for any $c > 0$, there exist a continuous increasing function $\lambda(\sigma^2)$ such that

$$\Pr_\sigma\Big(c < \frac{S^2}{k_n} < \lambda(\sigma^2)\Big) \le \frac{1}{2}, \quad \lambda(\sigma^2) > \sigma^2 > c,$$

$$\lambda(c) = c,$$

$$\Pr_\sigma\Big(\lambda(\sigma^2) < \frac{S^2}{k_n} < c\Big) \le \frac{1}{2}, \quad \lambda(\sigma^2) < \sigma^2 < c.$$

Figure 3 shows the boundary function $\mu_L(x)$ with the same loss function in Example 1 with $df = 10$ and $c = 1$. Then any continuous $T(X)$ satisfying (2.3) dominates X in the sense of GPCC.

Note. It is easily noted that if we define the boundary function

$$\mu_L(x) = \min(c, \lambda^{-1}(x)), \quad x \le c,$$

$$\mu_L(x) = \max(c, \lambda^{-1}(x)), \quad x \ge c,$$

then Proposition 1 holds with any loss function of the form $L(\cdot,\cdot)$.

3. COMPARISONS WITHIN THE DOMINATING CLASS

Returning now, for the sake of conciseness, to Piman's original closeness criterion, and to the case of real line support with $c = 0$, we now compare dominating estimators amongst each other.

To begin with, the second construction in David and Salem (1991) is easily extended to verify the claim that if

$$\lambda^{-1}(x) \le U_0(x) < U(x) < x, \quad x > 0, \tag{3.1a}$$

$$x < U(x) < U_0(x) \le \lambda^{-1}(x), \quad x < 0, \tag{3.1b}$$

then $U_0(X)$ Pitman-dominates $U(X)$. Thus, letting $\mathcal{S}$ be defined as in the Introduction, it may be claimed that an estimator $U_2(X)$ "near" $\lambda^{-1}(X)$ dominates more (in the sense related to (3.1)) members $U(X)$ of $\mathcal{S}$ than does an estimator $U_1(X)$ "near" X. We note that this claim is not in general extendable to the larger class of estimators $T(X)$ based on the boundary function $\mu(X)$.

Estimators $U_1(X)$ and $U_2(X)$ respectively "near" X and $\lambda^{-1}(X)$ may also be compared in regard to the behavior (particularly the tail behavior) of the random variable

$$Y_{U_2,U_1,\theta}(X) = |U_2(X) - \theta| - |U_1(X) - \theta|.$$

It is noted that the distribution of $Y_{U_2,U_1,\theta}$ tends to be skewed to the left (with smaller tails and more symmetry for large $|\theta|$).

A third comparison favors $U_1(X)$ over $U_2(X)$: Rao *et al.* (1986) defined Pitman nearness of $\hat{\theta}_2$ relative to $\hat{\theta}_1$ as the function of θ given by $\Pr_\theta(|\hat{\theta}_2 - \theta| < |\hat{\theta}_1 - \theta|)$. It may be verified that the Pitman nearness of $U_1(X)$ relative to X is greater than that of $U_2(X)$ relative to X, for all θ.

4. LOCATION-SCALE CASE WHEN THE SCALE PARAMETER IS BOUNDED BELOW

Let X be a sample from a member of a location-scale family of density functions $f(x;\theta,\sigma)$, with unknown median θ and unknown scale parameter σ. With σ seen as a nuisance parameter, in the sense of Sen *et al.* (1989), define $\Omega_0 = \{(\theta,\sigma) : -\infty < \theta < +\infty, \sigma \geq \sigma_0\}$.

Proposition 2. Let $\mathcal{S}_\sigma$ be the set of estimators satisfying (2.3), for the location problem in θ, with σ fixed, and let $T(X)$ be any member of $\mathcal{S}_{\sigma_0}$. Then

$$\Pr_{(\theta,\sigma)}(|T(X) - \theta| < |X - \theta|) > 1/2, \quad \forall (\theta,\sigma) \in \Omega_0. \tag{4.1}$$

Proof. It is readily verified that $\mathcal{S}_{\sigma_1} \subset \mathcal{S}_{\sigma_2}$, when $\sigma_1 < \sigma_2$. Hence, since $T(X)$ belongs to $\mathcal{S}_{\sigma_0}$, it belongs as well to any $\mathcal{S}_\sigma$, $\sigma > \sigma_0$.

Example 3. Let X be a single observation from the density

$$f(x;\theta,\sigma) = \frac{1}{2\sigma}\exp(-|x-\theta|/\sigma), \quad -\infty < x < +\infty$$

where θ is real valued unknown location parameter and σ, $\sigma \geq \sigma_0 > 0$, is unknown scale parameter. It is noted that X is a median-unbiased estimator of θ. Without loss of generality, we assume $\theta > 0$. It is easy to see that $\lambda^*(\theta) = \theta - \sigma_0 \ln(1 - e^{-\frac{\theta}{\sigma_0}})$ where $\lambda^*(\theta)$ satisfies

$$\int_0^{\lambda^*(\theta)} f(x;\theta,\sigma_0)\,dx = 1/2, \quad 0 < \theta < +\infty.$$

It is easy to see that $\lambda^*(x)$ is convex and has a unique minimum value $2\sigma_0 \ln 2$ at $x = \sigma_0 \ln 2$. Now take $\lambda(\theta)$ satisfying

$$\lambda(\theta) = \theta + \sigma_0 \ln(1 - e^{\frac{\theta}{\sigma_0}}), \qquad \theta < -\sigma_0 \ln 2,$$

$$\lambda(\theta) = 2\theta, \qquad -\sigma_0 \leq \theta \leq \sigma_0 \ln 2,$$

$$\lambda(\theta) = \theta - \sigma_0 \ln(1 - e^{-\frac{\theta}{\sigma_0}}), \qquad \theta > \sigma_0 \ln 2.$$

Then any continuous $T(X)$ satisfying (2.3) dominates X in the sense of GPCC with respect to the loss function of the form $L(\cdot,\cdot)$.

It is likely that Ω_0 in (4.1) cannot be replaced by $\Omega = \{(\theta,\sigma) : -\infty < \theta < +\infty, \sigma > 0\}$. Indeed, for large classes of estimators $T(X)$ with Lebesgue measure m of $\{x : T(x) \neq x\}$ greater than zero, there will be a θ_0 such that

$$\lim_{\sigma \to 0} \Pr_{(\theta_0,\sigma)}(|X - \theta| < |T(X) - \theta|) = 1;$$

for example, the class of $T(\cdot)$ with $T(x) \neq x$ for some x, and either (i) $T(\cdot)$ continuous or (ii) $T(\cdot)$ monotone. (Note that, in (i), it is sufficient that $T(\cdot)$ be equal to a continuous function on a set $R - E$ with $m(E) = 0$, and similary for (ii).)

ACKNOWLEDGEMENTS

We appreciate the helpful comments of Arnold Faden, Stephen Vardeman, and two unknown referees.

REFERENCES

(1) David, H.T. and Salem, A.S. (1991). Three Shrinkage Constructions for Pitman-Closeness in the One-dimensional Location Case. *Communications in Statistics*, **A20**, 3605–3627.

(2) Keating, J.P., Mason, R.L., and Sen, P.K. (1993). Pitman's Measure of Closeness: A Comparison of Statistical Estimators. *Society for Industrial and Applied Mathematics*, Philadelphia.

(3) Pitman, E.J.G. (1937). The "Closest" Estimates of Statistical Parameters. *Proceedings of the Cambridge Philosophical Society*, **33**, 212–222.

(4) Rao, C.R., Keating, J.P., and Mason, R.L. (1986). The Pitman Nearness Criterion and its Determination. *Communications in Statistics*, **A15**, 3173–3191.

(5) Robert, A.P., Hwang, J.T., and Strawderman, W.E. (1993). Is Pitman Closeness a Reasonable Criterion ? *Journal of the American Statistical Association*, **88**, 57–63.

(6) Sen, P.K., Kubokawa, T., and Saleh, A.K. (1989). The Stein Paradox in the Sense of the Pitman Measure of Closeness. *The Annals of Statistics*, **17**, 1375–1386.

(7) Yoo, S. and David, H.T. (1994). Shrinkage Constructions for Pitman Domination. *Sankhyā*, Ser. **B**, **56**, Part **1**, 107–114.