Journal of the Korean
Statistical Society
Vol. 24, No. 2, 1995

Asymptotic Properties of the Stopping Times in a Certain Sequential Procedure [†]

Sung Lai Kim [1]

ABSTRACT

In the problem of some sequential estimation, the stopping times may be written in the form $N(c) = \inf\{n \geq n_0; n \geq c^2 s_n^2/\delta^2(\overline{X}_n)\}$ where $\{s_n^2\}$ and $\{\overline{X}_n\}$ are the sequences of sample variance and sample mean of the independently and identically distributed (i.i.d.) random variables with distribution $F_\theta(x)$, $\theta \in \Theta$, respectively, and δ is either constant or any given positive real valued function. We obtain some asymptotic normality and asymptotic expectation of the $N(c)$ in various limiting situations. Specially, uniform asymptotic normality and uniform asymptotic expectation of the $N(c)$ are given.

KEYWORDS: Sequential estimation, Stopping times, β-protection, Uniform asymptotic normality, Uniform asymptotic expectation.

1. INTRODUCTION

Let $X, X_1, X_2, X_3, \ldots$ be independently and identically distributed random variables with distribution function F_θ, $\theta \in \Theta$. Let $\mu = \mu(\theta)$ and $\sigma^2 = \sigma^2(\theta)$ be the unknown mean and variance of the $F_\theta(x)$ respectively. Consider the problem of finding a confidence interval for $\mu(\theta)$ with β-protection at $\mu - \delta(\mu)$

[†]This paper was partly supported by NON DIRECTED RESEARCH FUND, Korea research foundation, 1991.

[1]Department of Mathematics, Chungnam National University, Taejon, 302–764, Korea.

and with probability of coverage at least α $(0 < \alpha < 1)$ for some imprecision function $\delta(x)$ which is either constant or positive real valued function having $\delta(x) \to 0$ as $x \to -\infty$. Then the stopping time of this sequential procedure is of the form.

$$N(c) = \inf\Big\{n \geq n_0 \ ; \ n \geq c^2 s_n^2/\delta^2(\overline{X}_n), \quad c > 0\Big\}, \tag{1.1}$$

where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$, $n \geq 2$.

In particular, if $\delta(x)$ is constant function, this stopping time is the same form of stopping times proposed by Chow and Robbins (1965), Starr (1966), Simons (1968), and Woodroofe (1977). For non-constant variable function $\delta(x)$, the stopping time proposed by Wijsman (1986) is of this form.

In this paper we derive the uniform asymptotic normality and the uniform asymptotic expectation of the stopping time as the parameter $c \to +\infty$ or $\sigma(\theta) \to +\infty$ in the some class of distribution when imprecision function $\delta(x)$ is constant. When $\delta(x)$ is variable function with $\delta(x) \to 0$ as $x \to -\infty$, under the some smoothness conditions of $\delta(x)$, we also derive the asymptotic distribution and the asymptotic expectation as $c \to +\infty$, and the uniform asymptotic distribution and the uniform asymptotic expectation in the some class of distributions as $\mu \to -\infty$.

Throughout this paper, we use the following notations;

$$Z_i = (X_i - \mu)/\sigma, \qquad \overline{Z}_n = \frac{1}{n}\sum_{n=1}^{n} Z_i,$$

$$s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2, \qquad s'^2_n = \frac{1}{n-1}\sum_{i=1}^{n}(Z_i - \overline{Z}_n)^2, \quad n \geq 2.$$

and the only restrictions on the family of distributions F_θ of random variable X are that they belong to the class

$$\mathfrak{F} = \Big\{F_\theta : \sup_{\theta\in\Theta} EZ^4 < B_0, \sup_{\theta\in\Theta} E|Z^2 - 1|^3/w^3(\theta) < D_0 \text{ for some } B_0, D_0 > 0\Big\},$$

where $w^2(\theta) = \mathrm{Var}(Z^2)$.

For the class $\mathfrak{F}$ uniform asymptotic normality and some uniform convergence results are derived easily using Berry-Esséen theorem and Kolmogorov's inequality, so we state the following lemma without proof.

Lemma 1.1. Let X_1, X_2, X_3, ..., be i.i.d. real valued random variables with distribution F_θ, $\theta \in \boldsymbol{\Theta}$ where $F_\theta \in \mathfrak{F}$. Then

(1) for each real number x

$$\lim_{n\to\infty} \sup_{F_\theta \in \mathfrak{F}} \left| P_\theta \left\{ \sum_{i=1}^{n} (Z_i^2 - 1)/\sqrt{n}w(\theta) \le x \right\} - \Phi(x) \right| = 0,$$

where Φ is the standard normal distribution function.

(2) For arbitrary given $\epsilon > 0$,

$$\sup_{F_\theta \in \mathfrak{F}} \sum_{n=1}^{\infty} P_\theta \left\{ \left| \sum_{i=1}^{n} Z_i \right| > n\epsilon \right\} < +\infty$$

and

$$\sup_{F_\theta \in \mathfrak{F}} \sum_{n=1}^{\infty} P_\theta \left\{ \left| \sum_{i=1}^{n} (Z_i^2 - 1) \right| > n\epsilon \right\} < +\infty.$$

(3) For $\alpha > \frac{1}{2}$, $\sum_{i=1}^{n} (Z_i^2 - 1)/n^\alpha \to 0$ a.s. as $n \to \infty$ uniformly in $\mathfrak{F}$.

Let $Y, Y_1, Y_2, Y_3, \ldots$ be i.i.d. real valued random variables with distribution F_θ, $\theta \in \boldsymbol{\Theta}$, where $F_\theta \in \mathfrak{F}^*$, some class of distributions. Let F_θ have a finite positive mean $\nu = \nu(\theta)$ and finite variance $\tau^2 = \tau^2(\theta)$. Let $\{\xi_n : n \ge 1\}$ be real valued random variables for which $(Y_i, \xi_i)_{i=1}^n$ are independent of Y_k, $k > n$, for all $n \ge 1$. Put $W_n = U_n + \xi_n$, where $U_n = \sum_{i=1}^{n} Y_i$. Define, for all $a > 0$, $N_a = a/\nu$, and $t_a = \inf\{n \ge 1 : W_n > a\}$.

The following two Lemmas are modified from Woodroofe (1982). The proofs are similar to those of Woodroofe, so we omit the proof.

Lemma 1.2. Assume that for some $\eta > 0$, there exists $k > 0$ such that $E_\theta |Y/\tau|^{2(1+\eta)} < k$ uniformly in $\mathfrak{F}^*$. If (i) $(U_n - n\nu)/\sqrt{n}\tau \xrightarrow{\mathfrak{L}} N(0,1)$ as $n \to \infty$

uniformly in $\mathfrak{F}^*$, (ii) $\xi_{t_a}/(\tau N_a^{\frac{1}{2}}) \xrightarrow{p} 0$ as $a \to \infty$ uniformly in $\mathfrak{F}^*$, and (iii) $t_a/N_a \xrightarrow{p} 1$ as $a \to +\infty$ uniformly in $\mathfrak{F}^*$. Then $\frac{\nu}{\tau}(t_a - N_a)/N_a^{\frac{1}{2}} \xrightarrow{\mathfrak{L}} N(0,1)$ as $a \to \infty$ uniformly in $\mathfrak{F}^*$.

Lemma 1.3. Suppose $\nu(\theta)$ is a constant and for arbitary given $\epsilon > 0$, $\delta > 0$ such that $\epsilon + \delta < \nu$. If (i) both $\sum_{n=1}^{\infty} P_\theta(U_n - n\nu \leq -n\delta)$ and $\sum_{n=1}^{\infty} P_\theta(\xi_n \leq -n\epsilon)$ converge uniformly in $\mathfrak{F}^*$ and (ii) $t_a/N_a \xrightarrow{p} 1$ as $a \to \infty$ uniformly in $\mathfrak{F}^*$. Then $E_\theta\, t_a/N_a \to 1$ as $a \to \infty$ uniformly in $\mathfrak{F}^*$.

2. MAIN RESULTS

In this section we obtain some asymptotic properties of the stopping time $N(c)$ defined as in (1.1).

First, when $\delta(x)$ is constant, i.e., $\delta(x) = 1$ for all $x \in \mathbb{R}$, then we can rewrite the stopping time $N(c)$ given by (1.1) in the form

$$N(c) = \inf\Big\{n \geq n_0 \; ; \; U_n + \xi_n \geq c^2\sigma^2\Big\}, \tag{2.1}$$

where $U_n = \sum_{i=1}^{n}(2 - Z_i^2)$ and $\xi_n = n\Big(1/s'^2_n - 2 + \frac{1}{n}\sum_{i=1}^{n} Z_i^2\Big)$.

Theorem 2.1. Let $N(c)$ defined as in (2.1) and set $N_0 = c^2\sigma^2$. Then for arbitary given $\sigma_0 > 0$ and $w_0 > 0$

(1) $(N(c) - N_0)/N_0^{\frac{1}{2}} w(\theta) \xrightarrow{\mathfrak{L}} N(0,1)$ as $c \to \infty$ uniformly in $\mathfrak{F}_1$, where $\mathfrak{F}_1 = \{F_\theta \in \mathfrak{F} : \sigma(\theta) \geq \sigma_0,\; w(\theta) \geq w_0\}$.

(2) For any given $c > 0$, $(N(c) - N_0)/N_0^{\frac{1}{2}} w(\theta) \xrightarrow{\mathfrak{L}} N(0,1)$ as $\sigma(\theta) \to +\infty$ uniformly in $\mathfrak{F}_2$, where $\mathfrak{F}_2 = \{F_\theta \in \mathfrak{F} : \sigma(\theta) = \sigma,\;\; w(\theta) \geq w_0\}$.

Proof. In Lemma 1.2 take $Y_i = 2 - Z_i^2$ so that $\tau = w$ and take $\eta = \dfrac{1}{2}$. Then $E_\theta|Y/w(\theta)|^3 \leq D_0 + 3(w_0^{-1} + w_0^{-2}) + w_0^{-3}$ since $w \geq w_0$. By Lemma 1.1, assumption (i) in Lemma 1.2 holds and we can easily prove that $N/N_0 \to 1$ a.s. as either $c \to \infty$ uniformly in $\mathfrak{F}_1$ or as $\sigma(\theta) \to +\infty$ for any given $c > 0$ uniformly in $\mathfrak{F}_2$. It remains to show that $\xi_N/w(\theta)N_0^{\frac{1}{2}}$ converges uniformly to zero in probability in the two cases. Since $w(\theta) \geq w_0$, it is enough to show $\xi_N/N_0^{\frac{1}{2}}$ converges uniformly to zero in probability in the two cases. Using Taylor's expansion

$$\xi_n = -\frac{1}{n-1}\sum_{i=1}^{n} Z_i^2 + \frac{n^2}{n-1}\overline{Z}_n^2 + n(V_n)^{-3}(s'^2_n - 1)^2, \quad \text{where } V_n \in [s'^2_n, 1].$$

We can easily show that both $\frac{1}{\sqrt{n}} \cdot \frac{1}{n-1}\sum_{i=1}^{n} Z_i^2$ and $\frac{1}{\sqrt{n}} \cdot \frac{n^2}{n-1}\overline{Z}_n^2$ converge to zero in probability as $n \to \infty$ uniformly in $\mathfrak{F}$ and $n(V_n)^{-3}(s'^2_n - 1)^2/\sqrt{n} = V_n^{-3}\left\{\frac{n}{n-1}\sum_{i=1}^{n}\frac{Z_i^2 - 1}{n^{3/4}} + n^{\frac{1}{4}}/(n-1) - \frac{n}{n-1}\left(\sum_{i=1}^{n} Z_i/n^{7/8}\right)^2\right\}^2 \to 0$ a.s. as $n \to \infty$ uniformly in $\mathfrak{F}$ by Lemma 1.1. Thus we have shown that $\xi_n/\sqrt{n}$ converges to zero a.s. as $n \to \infty$ uniformly in $\mathfrak{F}$. So $\xi_N/N_0^{\frac{1}{2}}$ converges to zero a.s. in the two cases uniformly in $\mathfrak{F}$ by Woodroofe (1982, p.41) and Anscombe (1952).

Theorme 2.2. Let $N(c)$ and N_0 be as in Theorem 2.1. Then

(1) $E_\theta\ N(c)/N_0 \to 1$ as $c \to \infty$ uniformly in $\mathfrak{F}_1$, where $\mathfrak{F}_1 = \{F_\theta \in \mathfrak{F}\ ;\ \sigma(\theta) \geq \sigma_0\}$ for any given $\sigma_0 > 0$.

(2) $E_\theta\ N(c)/N_0 \to 1$ as $\sigma(\theta) \to \infty$ uniformly in $\mathfrak{F}_2$, where $\mathfrak{F}_2 = \{F_\theta \in \mathfrak{F} : \sigma(\theta) = \sigma\}$ for any given $c > 0$.

Proof. For arbitary given $\epsilon_1 > 0$, $\sum_{n=k}^{\infty} P_\theta\{S_n - E_\theta S_n \leq -n\epsilon_1\} \leq \sum_{n=k}^{\infty} P_\theta\left\{\left|\sum_{i=1}^{n}(Z_i^2 - 1)\right| \geq n\epsilon_1\right\} \to 0$ as $k \to \infty$ uniformly in $\mathfrak{F}$ by Lemma 1.1 and also for arbitary given $\epsilon_2 > 0$, $\sum_{n=k}^{\infty} P_\theta\{\xi_n \leq -n\epsilon_2\} \leq \sum_{n=k}^{\infty} P_\theta\left\{\sum_{i=1}^{n} Z_i^2 \geq n(n-1)\epsilon_2\right\} \leq B_0\sum_{n=k}^{\infty}\frac{1}{n(n-1)^2\epsilon_2^2} + \sum_{n=k}^{\infty}\frac{1}{n(n-1)\epsilon_2^2} \to 0$ as $k \to \infty$ uniformly in $\mathfrak{F}$. Since S_n

is the sum of i.i.d. random variables with mean 1, take the ϵ_i $(i = 1, 2)$ above so that $\epsilon_1 + \epsilon_2 < 1$. So Theorem follows by Lemma 1.3 since $N(c)/N_0 \to 1$ a.s. as either $c \to \infty$ uniformly in $\mathfrak{F}_1$ or $\sigma(\theta) \to 0$ uniformly in $\mathfrak{F}_2$.

Second, the imprecision function $\delta(x)$ is not constant, we make following smoothness assumption of $\delta(x)$.

(i) $\delta(x)$ is positive bounded with $\delta(x) \to 0$ as $x \to -\infty$.

(ii) $\delta(x)$ is differentiable with $0 < \delta'(x) < M$ for some $M < +\infty$.

(iii) $\delta(x + y)/\delta(x) \to 1$ as $y \to 0$ uniformly in x, and

(iv) For any given $\epsilon > 0$, $B > 0$, $0 < \xi \leq 1$, there exists x_0 such that $(1 + \epsilon)\delta(x) - \delta(x + B/(\delta(x))^{\xi}) > 0$ for all $x \leq x_0$.

The following two Lemmas follow easily from Chow and Robbins (1965) or Kim (1990).

Lemma 2.3. Let $N(c)$ be defined by (1.1) and set $N_0 = c^2\sigma^2/\delta^2(\mu)$. Then under the smoothness conditions about $\delta(x)$, for any given $\theta \in \boldsymbol{\Theta}$,

(1) $N(c) \to \infty$ a.s. as $c \to \infty$.

(2) $N(c)/N_0 \to 1$ a.s. as $c \to \infty$.

Lemma 2.4. Let $N(c)$ and N_0 be as in Lemma 2.3. Then, for any fixed $c > 0$, under the smoothness conditions about $\delta(x)$,

(1) $N(c) \to +\infty$ a.s. as $\mu \to -\infty$ uniformly in $\mathfrak{F}_1$.

(2) $N(c)/N_0 \to 1$ a.s. as $\mu \to -\infty$ uniformly in $\mathfrak{F}_1$,

where $\mathfrak{F}_1 = \{F_\theta \in \mathfrak{F} : \mu(\theta) = \mu, \sigma(\theta) = \sigma\}$.

Under the smoothness conditions about $\delta(x)$, we have the following asymptotic normality and asymptotic expectation for any given $\theta \in \boldsymbol{\Theta}$.

Theorem 2.5. Let $N(c)$ be defined in (1.1) and set $N_0 = c^2\sigma^2/\delta^2(\mu)$. Then $(N(c) - N_0)/N_0^{\frac{1}{2}} \xrightarrow{\mathfrak{L}} N(0,q)$ as $c \to \infty$ for any given $\theta \in \Theta$, where $q = E_\theta Z^4 - 1 + 4\delta^{-2}(\mu)\delta'^2(\mu)\sigma^2 - 4\delta^{-1}(\mu)\delta'(\mu)\sigma E_\theta Z^3$.

Proof. Rewrite the stopping time $N(c)$ defined in (1.1) in the form

$$N(c) = \inf\Big\{n \geq n_0 : S_n + \xi_n \geq c^2\sigma^2\Big\}, \tag{2.2}$$

where

$$\begin{aligned} S_n &= \sum_{i=1}^{n} \left\{\delta^2(\mu)(2 - Z_i^2) + 2\delta(\mu)\delta'(\mu)\sigma Z_i\right\} \\ \xi_n &= n\left\{\delta^2(\mu + \sigma\overline{Z}_n) - \delta^2(\mu) - 2\delta(\mu)\delta'(\mu)\sigma\overline{Z}_n\right\}/s'^2_n \\ &\quad + 2n\delta(\mu)\delta'(\mu)(s'^2_n - 1)\sigma\overline{Z}_n + n\delta^2(\mu)\Big(s'^2_n - 2 + \frac{1}{n}\sum_{i=1}^{n} Z_i^2\Big). \end{aligned}$$

Then S_n is the sum of i.i.d random variable with mean $\delta^2(\mu)$, variance $q\delta^4(\mu)$. It is enough to show that $\xi_{N(c)}/N_0^{\frac{1}{2}} \to 1$ in probability as $c \to \infty$ since $N(c)/N_0 \to 1$ a.s. as $c \to \infty$ by Lemma 2.3. Put $\xi_n = \xi_n^{(1)} + \xi_n^{(2)} + \xi_n^{(3)}$, where

$$\begin{aligned} \xi_n^{(1)} &= n\left\{\delta^2(\mu + \sigma\overline{Z}_n) - \delta^2(\mu) - 2\delta(\mu)\delta'(\mu)\sigma\overline{Z}_n\right\}/s'^2_n \\ \xi_n^{(2)} &= 2n\delta(\mu)\delta'(\mu)(s'^2_n - 1)\sigma\overline{Z}_n \\ \xi_n^{(3)} &= n\delta^2(\mu)\Big(s'^2_n - 2 + \frac{1}{n}\sum_{i=1}^{n} Z_i^2\Big). \end{aligned}$$

Then

$$\xi_n^{(1)}/\sqrt{n} = 2\sigma\sqrt{n}\overline{Z}_n\left\{\delta(\nu_n)\delta'(\nu_n) - \delta(\mu)\delta'(\mu)\right\}/s'^2_n, \text{ where } |\nu_n - \mu| \leq |\sigma\overline{Z}_n|.$$

$\{2\sigma(\theta)\left(\delta(\nu_n)\delta'(\nu_n) - \delta(\mu)\delta'(\mu)\right)/s'^2_n\}$ is u.c.i.p (uniformly continuity in probability) and $\{\sqrt{n}\overline{Z}_n\}$ is also u.c.i.p. Therefore $\xi_n^{(1)}/\sqrt{n}$ is u.c.i.p by Woodroofe (1982). Similarly we can prove that $\xi_n^{(2)}/\sqrt{n} = 2\sqrt{n}\delta(\mu)\delta'(\mu)(s'^{-2}_n - 1)\sigma\overline{Z}_n$ and $\xi_n^{(3)}/\sqrt{n} = \sqrt{n}\delta^2(\mu)\Big(s'^{-2}_n - 2 + \frac{1}{n}\sum_{i=1}^{n} Z_i^2\Big)$ are u.c.i.p. Therefore $\{\xi_n/\sqrt{n} : n \geq 2\}$ is u.c.i.p. We can easily prove that $\xi_n/\sqrt{n} \to 0$ in probability as $n \to \infty$, so the theorem follows from Lemma 4.2 of Woodroofe (1982).

Theorem 2.6. Let $N(c)$ and N_0 be as in Theorem2.5. Then

$$E_\theta N(c)/N_0 \to 1 \text{ as } c \to \infty \text{ for any given } \theta \in \mathbf{\Theta}.$$

Proof. Let S_n and ξ_n be as in the proof of Theorem 2.5. Since $N(c)/N_0 \to 1$ in probability as $c \to \infty$ for any given $\theta \in \mathbf{\Theta}$ and for arbitary given $\epsilon > 0$, $\theta \in \mathbf{\Theta}$, $\sum_{n=k}^{\infty} P_\theta\{S_n - ES_n \le -n\epsilon\} \le \sum_{n=k}^{\infty} P_\theta\Big\{\Big|\sum_{i=1}^{n}(Z_i^2 - 1)\Big| \ge n\epsilon/2\delta^2(\mu)\Big\} + \sum_{n=k}^{\infty} P_\theta\Big\{\Big|\sum_{i=1}^{n} Z_i\Big| \ge n\epsilon/(4\delta(\mu)\delta'(\mu)\sigma)\Big\} \to 0$ as $k \to \infty$ by Lemma 1.1. It is enough to show that $\sum_{n=2}^{\infty} P_\theta\{\xi_n \le -n\epsilon\}$ converges for some $0 < \epsilon < \delta^2(\mu)$ by Theorem 4.4 of Woodroofe (1982). If we put $\xi_n^{(i)}$, $i = 1, 2, 3$, as in the proof of Theorem 2.5, then $\sum_{n=2}^{\infty} P_\theta\{\xi_n \le -n\epsilon\} \le \sum_{i=1}^{3}\{\sum_{n=2}^{\infty} P_\theta(\xi_n^{(i)} \le -n\epsilon/3)\}$. For $i = 1$, $\sum_{n=2}^{\infty} P_\theta(\xi_n^{(1)} \le -n\epsilon/3) \le \sum_{n=2}^{\infty} P_\theta(s_n'^2 < 1 - \epsilon) + \sum_{n=2}^{\infty} P_\theta\{2\sigma\overline{Z}_n(\delta(\nu_n)\delta'(\nu_n) - \delta(\mu)\delta'(\mu)) \le -\epsilon(1-\epsilon)/3\}$, where $|\nu_n - \mu| \le |\sigma\overline{Z}_n|$. But

$$\sum_{n=2}^{\infty} P_\theta\Big(s_n'^2 < 1 - \epsilon\Big) \le \sum_{n=2}^{\infty} P_\theta\Big\{\sum_{i=1}^{n}(Z_i^2 - 1) \le -n\epsilon/2\Big\} + \sum_{n=2}^{\infty} P_\theta\Big\{\Big|\sum_{i=1}^{n} Z_i\Big| \ge n(\epsilon/2)^{\frac{1}{2}}\Big\} < +\infty$$

by Lemma 1.1 again and

$$\sum_{n=2}^{\infty} P_\theta\Big\{2\sigma\overline{Z}_n(\delta(\nu_n)\delta'(\nu_n) - \delta(\mu)\delta'(\mu)) \le -\epsilon(1-\epsilon)/3\Big\} \le \sum_{n=2}^{\infty} P_\theta\Big\{\Big|\sum_{i=1}^{n} Z_i\Big| \ge n\epsilon(1-\epsilon)/(12\sigma LM)\Big\} < +\infty$$

since δ and δ' are bounded by $L(say)$ and M respectively, $|\delta(\nu_n)\delta'(\nu_n) - \delta(\mu)\delta'(\mu)| \le 2LM$. Similarly $\sum_{n=2}^{\infty} P_\theta(\xi_n^{(i)} \le -n\epsilon/3) < +\infty$ can be proved for $i = 1, 2$.

We will now show that asymptotic properties of the stopping time $N(c)$ defined as in (1.1) as $\mu \to -\infty$ for any fixed $c > 0$ and $\sigma > 0$ under stronger conditions on $\delta(x)$ than specified smoothness conditions on $\delta(x)$. If the smoothness condition (iii) about $\delta(x)$ is replaced by the stronger condition (iii)$'$; For some $b > 0$, $(\delta(x+y) - \delta(x))/y\delta(x) \to 0$ as $x \to -\infty$ uniformly in $|y| \le b$, then (iii)$'$ implies (iii) and $|(\delta(x+y)-\delta(x))/y\delta(x)|$ is bounded, say, R_0, for all $|y| \le b$ and for all x.

Theorem 2.7. Let $N(c)$ and N_0 be as in Theorem 2.5. Let $c > 0$ and $\sigma > 0$ be fixed and let $w_0 > 0$ be arbitary. Then under the stronger smoothness conditions on $\delta(x)$.

(1) $(N(c) - N_0)/N_0^{\frac{1}{2}} w(\theta) \xrightarrow{\mathcal{L}} N(0,1)$ as $\mu \to -\infty$ uniformly in $\mathfrak{F}_1$, where $\mathfrak{F}_1 = \{F_\theta \in \mathfrak{F} : w(\theta) \ge w_0, \quad \mu(\theta) = \mu, \quad \sigma(\theta) = \sigma\}$.

(2) $E_\theta N(c)/N_0 \to 1$ as $\mu \to -\infty$ uniformly in $\mathfrak{F}_2$, where

$$\mathfrak{F}_2 = \{F_\theta \in \mathfrak{F} : \mu(\theta) = \mu, \quad \sigma(\theta) = \sigma\}.$$

Proof. We can write $N(c)$ as in (1.1) in the form

$$N(c) = \inf\Big\{n \ge 2 : S_n + \xi_n \ge c^2\sigma^2/\delta^2(\mu)\Big\},$$

where $S_n = \sum_{i=1}^{n}(2 - Z_i^2)$, $\xi_n = n\Big(s_n'^{-2} - 2 + \frac{1}{n}\sum_{i=1}^{n} Z_i^2\Big) + n\Big\{\delta^2(\mu + \sigma\overline{Z}_n) - \delta^2(\mu)\Big\}\Big/(\delta^2(\mu)s_n'^2)$. By Lemma 2.4 $N(c)/N_0 \to 1$ a.s. as $\mu \to -\infty$ uniformly in $\mathfrak{F}_2$. For the proof of (1), it suffices to show that $\xi_{N(c)}/N_0^{\frac{1}{2}}$ converges to zero in probability as $\mu \to -\infty$ uniformly in $\mathfrak{F}_1$ since $E_\theta|(Z^2-2)/w(\theta)|^3$ is bounded when $w(\theta) \ge w_0$ in the proof of Theorem 2.1. Put $\xi_n = \xi_n^{(1)} + \xi_n^{(2)}$, where $\xi_n^{(1)} = n(s_n'^2 - 2 + \frac{1}{n}\sum_{i=1}^{n} Z_i^2)$ and $\xi_n^{(2)} = n\{\delta^2(\mu + \sigma\overline{Z}_n) - \delta^2(\mu)\}/(\delta^2(\mu)s_n'^2)$, then $\xi_{N(c)}^{(1)}/N_0^{\frac{1}{2}} = (N(c)/N_0)^{\frac{1}{2}}(\xi_{N(c)}^{(1)}/N(c)^{\frac{1}{2}}) \to 0$ in probability as $\mu \to -\infty$ uniformly in $\mathfrak{F}_1$.

Observe that for any given $c > 0$, as $\mu \to -\infty$ $(N(c)/N_0)^{\frac{1}{2}}\{\delta(\mu + \sigma\overline{Z}_n) + \delta(\mu)\}/\delta(\mu)s_n'^2 \to 2$ in probability uniformly in $\mathfrak{F}_1$ and $(\delta(\mu + \sigma\overline{Z}_n) -$

$\delta(\mu))/(\sigma\overline{Z}_n\delta(\mu)) \to 0$ a.s. by smoothness condition (iii)′ since $\sigma\overline{Z}_n \to 0$ a.s. uniformly in $\mathfrak{F}_1$, so that eventually $|\sigma\overline{Z}_n| \leq b$ for some $b > 0$. Therefore $\xi^{(2)}_{N(c)}/N_0^{\frac{1}{2}} \to 0$ in probability as $\mu \to -\infty$ for any given $c > 0$ uniformly in $\mathfrak{F}_1$.

For the proof of (2), it suffices to show that for arbitary $0 < \epsilon < 1$, $\sum_{n=k}^{\infty} P_\theta\{\xi_n < -n\epsilon\} \to 0$ as $k \to \infty$ uniformly in $\mathfrak{F}_2$. But

$$\sum_{n=k}^{\infty} P_\theta\Big\{\xi_n \leq -n\epsilon\Big\} \leq \sum_{n=k}^{\infty} P_\theta\Big\{s'^2_n - 2 + \frac{1}{n}\sum_{i=1}^{n} Z_i^2 < -\epsilon/2\Big\} \tag{2.3}$$
$$+ \sum_{n=k}^{\infty} P_\theta\Big\{(\delta^2(\mu+\sigma\overline{Z}_n) - \delta^2(\mu))/(\delta^2(\mu)s'^2_n) \leq -\epsilon/2\Big\}$$

The first term on the right-hand side of (2.3) goes to 0 as $k \to \infty$ uniformly in $\mathfrak{F}_2$. By smoothness conditions (i) and (ii) on $\delta(x)$, the second term on the right-hand side of (2.3) can be bounded from above by

$$\sum_{n=k}^{\infty} P_\theta \Big\{2n(\delta(\mu+\sigma\overline{Z}_n) - \delta(\mu))/(\delta(\mu)s'^2_n) \leq -n\epsilon/2, |\sigma\overline{Z}_n| \leq b\Big\} \tag{2.4}$$
$$+ \sum_{n=k}^{\infty} P_\theta\Big\{\sigma\overline{Z}_n \leq -b\Big\} \text{ for any chosen } b > 0.$$

The second term in (2.4) converges to 0 as $k \to \infty$ uniformly in $\mathfrak{F}_2$ and take b and R_0 so that $|(\delta(\mu+\sigma\overline{Z}_n) - \delta(\mu))/(\sigma\overline{Z}_n\delta(\mu))| < R_0$ for all $|\sigma\overline{Z}_n| \leq b$ and for all μ, then the first term of (2.4) is bounded above by $\sum_{n=k}^{\infty} P_\theta\{|\overline{Z}_n| > \epsilon(1-\epsilon)/4\sigma R_0\} + \sum_{n=k}^{\infty} P_\theta(s'^2_n < 1-\epsilon)$ which goes to 0 as $k \to \infty$ uniformly in $\mathfrak{F}_2$.

REFERENCES

(1) Anscombe, F.J. (1952). Large Sample Theory of Sequential Estimation. *Proceedings of the Cambridge Philosophical Society*, **48**, 600–607.

(2) Chow, Y.S. and Robbins, H. (1965). On the Asymptotic Theory of Fixed-width Sequential Confidence Intervals for the Mean. *The Annals of Mathematical Statistics*, **36**, 457–462.

(3) Kim, S.L. (1990). Sequential Confidence Intervals for the Mean with β-protection in a Certain Parameter Space. *Journal of the Korean Statistical Society*, **19**, 113–121.

(4) Simons, G. (1968). On the Cost of Not Knowing the Variance when Makig a Fixed-width Conf the Mean. *The Annals of Mathematical Statistics*, **39**, 1946–1952.

(5) Sriram, T.N., Basawa, I.V., and Huggins, R.M. (1991). Sequential Estimation for Branching Processes with Immigration. *The Annals of Statistics*, **19**, 2232–2243.

(6) Starr, N. (1966). The Performance of a Sequential Procedure for the Fixed-width Interval Estimation of the Mean. *The Annals of Mathematical Statistics*, **37**, 36–47.

(7) Wijsman, R.A. (1981). Confidence Sets Based on Sequential Tests. *Communications in Statistics - Theory and Methods*, **A 10**, 2137–2147.

(8) Wijsman, R.A. (1983). Sequential Estimation, and Test-Oriented Confidence Intervals for μ/σ in a Normal Population. *Sequential Analysis*, **2**, 149–174.

(9) Wijsman, R.A. (1986). Sequential Confidence Intervals with Beta Protection in One-parameter Families. *Adaptive Statistical Procedures and Related Topics*, 96–107.

(10) Woodroofe, M. (1977). Second Order Approximations for Sequential Point and Inteval Estimation. *The Annals of Statistics*, **5**, 985–995.

(11) Woodroofe, M. (1982). *Nonlinear Renewal Theory in Sequential Analysis*. CBMS-NSF Regional Conference SIAM.

(12) Yoshikazu, T. (1992). A Sequential Procedures with Asymptotically Negative Regret for Estimating a Normal Mean. *The Annals of Statistics*, **20**, 562–569.