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A Weak Convergence Theorem for Mixingale Arrays

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ABSTRACT

This paper gives a generalization of an L_1-convergence theorem for dependent processes due to Andrews(1988) and also a probability convergence theorem.

KEYWORDS: Weak convergence, Mixingale array, Uniformly integrable in the Cesàro sense.

1. INTRODUCTION

Andrew(1988) combines a theorem on martingale convergence due to Chow (1971) with techniques developed by McLeish(1975a, 1975b, 1977) to obtain a law of large numbers for mixingales. Davidson(1993) extends these results to allow for global heterogeneity, including cases where the moments of a sequence are tending to either infinity or zero.

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Gut(1992) and Chandra(1989) provide a fairly general weak law for arrays using a domination condition, namely, uniform integrability in the Cesàro sense(UIC). The purpose of this note is to extend Andrews' result to similar type of mixingale arrays in Davidson(1993) satisfying condition (UIC). We also consider a convergence in probability under similar type of domination condition.

Let an array of pairs $\{X_{nt}, \mathcal{F}_{nt}; -\infty \leq t \leq \infty, n \geq 1\}$ be defined on a probability space $(\Omega, \mathcal{F}, P)$, where the X_{nt} are random variables and the $\mathcal{F}_{nt}$ are σ-subfields of $\mathcal{F}$, increasing in T. The array will be called an L_p-mixingale for $p \geq 1$, if there exists an array of nonnegative constants $\{c_{nt}\}$, and also a nonnegative sequence $\{\zeta_m\}_0^\infty$ such that $\zeta_m \to 0$ as $m \to \infty$, and

$$\|E(X_{nt}|\mathcal{F}_{n,t-m})\|_p \leq c_{nt}\zeta_m, \tag{1.1}$$

$$\|X_{nt} - E(X_{nt}|\mathcal{F}_{n,t+m})\|_p \leq c_{nt}\zeta_{m+1}, \tag{1.2}$$

holds for all t, n and $m \geq 0$. The sequence $\{\zeta_m\}$ is sometimes said to be of *size* λ_0 if $\zeta_m = O(m^{-\lambda})$ for $\lambda > \lambda_0$. In the case where $\zeta_m = 0$ for $m > 0$, the array becomes a martingale difference (m.d.). The single-indexed case where $X_{nt} = X_t, \mathcal{F}_{nt} = \mathcal{F}_i$ and $c_{nt} = c_t$ for each n will be called a mixingale sequence.

An array $\{X_{nt}, \mathcal{F}_{nt}\}$ of random variables is uniformly integrable in the Cesàro sense if

$$\sup_{n,i} \frac{1}{k_n} \sum_{t=i}^{i+k_n-1} E|X_{nt}|I(|X_{nt}| > a) \to 0 \text{ as } a \to \infty.$$

It is clear that uniform integrability is stronger than uniform integrability in the Cesàro sense (see e.g. Chandra (1989, Example 2)).

The main results are the following.

Theorem 1. Let the array $\{X_{nt}, \mathcal{F}_{nt}\}$ be a L_1-mixingale with respect to constant array of $\{c_{nt}\}$ such that

(a) $\limsup_{n\to\infty} \sup_{1\leq t\leq k_n} c_{nt}k_n < \infty$,

(b) $\limsup \sum_{t=1}^{k_n} c_{nt}^2 = 0$,

(c) $\{X_{nt}/c_{nt}\}$ is uniformly integrable in the Cesàro sense,

where k_n is an increasing, integer-valued function of n and $k_n \to \infty$ as $n \to \infty$. Then

$$\lim_{n\to\infty} E\Big|\sum_{t=1}^{k_n} X_{nt}\Big| = 0.$$

This is a very general result for which some special cases are more familiar then others. The case where $X_{nt} = X_t/a_n$ where $\{a_n\}$ is a positive constant sequence, and $\mathcal{F}_{nt} = \mathcal{F}_t$, each n, is important enough to deserve stating as a corollary.

Corollary 1. Suppose $\{X_t, \mathcal{F}_t\}_1^\infty$ is a L_1-mixingale sequence with respect to constant sequence $\{b_t\}_1^\infty$, and $\{a_n\}_1^\infty$ is another positive constant sequence such that

(a) $\sup_{i\leq n} b_i n = O(a_n)$,

(b) $\sum_{t=1}^{n} b_t^2 = o(a_n^2)$,

(c) $\{X_t/b_t\}$ is uniformly integrable in the Cesàro sense.

Then

$$\lim_{n\to\infty} E\Big|a_n^{-1}\sum_{t=1}^{n} X_t\Big| = 0.$$

It is easily verified that the conditions are observed when $b_t = t^\alpha$ for any $\alpha \geq -1$, by choosing $a_n = n^{1+\alpha}$ for $\alpha > -1$, and $a_n = \log n$ for $\alpha = -1$. In particular, when $b_t = 1$ for all t and $a_n = n$ we have the result that for a uniformly integrable in the Cesàro sense L_1-mixingale of arbitrary size, $\lim_{n\to\infty} E|X_n| = 0$ where $X_n = n^{-1}\sum_{t=1}^{n} X_t$. This extends Andrews' (1988) Theorem 1.

Let $X'_{nt} = X_{nt}I(|X_{nt}/c_{nt}| \leq k_n)$, where k_n is an increasing positive integer-valued function of n and $k_n \to \infty$ as $n \to \infty$.

Theorem 2. Let the array $\{X'_{nt}, \mathcal{F}_{nt}\}$ be an L_1-mixingale with respect to constant array of $\{c_{nt}\}$ such that

(a) $\limsup_{n\to\infty} \sup_{1\leq t\leq k_n} c_{nt}k_n < \infty$,

(b) $\limsup_{n\to\infty} \sum_{t=1}^{k_n} c_{nt}^2 = 0$,

(c) $\lim_{a\to\infty} \sup_{n,i} \frac{1}{k_n} \sum_{t=i}^{i+k_n-1} aP\{|\frac{X_{nt}}{c_{nt}}| > a\} = 0.$

Then

$$\sum_{t=1}^{k_n} X_{nt} \to 0 \text{ in probability as } n \to \infty.$$

Remark 1. Although Theorem 1 has a stronger domination condition than Theorem 2, Theorem 1 is stronger result than Theorem 2 in that Theorem 1 shows L_1 convergence(which implies convergence in probability).

2. PROOFS

We need the following lemma to prove Theorem 1.

Lemma 1. Suppose that $\{X_{nt}, \mathcal{F}_{nt}\}$ is a m.d. array satisfying conditions (a) and (b) of Theorem 1, and where the array $\{|X_{nt}/c_{nt}|^p\}$ is uniformly integrable in the Cesàro sense, $1 \leq p < 2$. Then

$$\lim_{n\to\infty} \Big\| \sum_{t=1}^{k_n} X_{nt} \Big\|_p = 0.$$

Proof. Let $M > 0$ and set, for $1 \leq t \leq k_n$, $n \geq 1$, $Y_{nt} = X_{nt}I(|X_{nt}| \leq Mc_{nt})$ and $Z_{nt} = X_{nt}I(|X_{nt}| > Mc_{nt})$, so that $X_{nt} = Y_{nt} + Z_{nt}$. With the aid of Burkholder's (1966) and Davis' (1970) inequality ($1 < p < 2$ and $p = 1$, respectively) and the c_r-inequality (Loéve (1977, p.157)) we obtain

$$\begin{aligned} E\Big|\sum_{t=1}^{k_n} X_{nt}\Big|^p &\le B_p E\Big|\sum_{t=1}^{k_n}(X_{nt})^2\Big|^{p/2} \\ &\le B_p E\Big|\sum_{t=1}^{k_n}(Y_{nt})^2\Big|^{p/2} + B_p E\Big|\sum_{t=1}^{k_n}(Z_{nt})^2\Big|^{p/2} \\ &\le B_p M^p\Big(\sum_{t=1}^{k_n} c_{nt}^2\Big)^{p/2} + B_p E\sum_{t=1}^{k_n}\Big|Z_{nt}\Big|^p \\ &\le B_p M^p\Big(\sum_{t=1}^{k_n} c_{nt}^2\Big)^{p/2} + B_p C k_n^{-1}\sum_{t=1}^{k_n} E\Big|\frac{X_{nt}}{c_{nt}}\Big|^p I\Big\{\Big|\frac{X_{nt}}{c_{nt}}\Big| > M\Big\}. \end{aligned}$$

Here B_p is a numerical constant, depending only on p and C is a positive constant. At first letting $n \to \infty$ and then letting $M \to \infty$ the conclusion follows.

Proof of Theorem 1. If $\{X_{nt}\}$ is uniformly integrable in the Cesàro sense it is easy to see that $\{E(X_{nt}|\mathcal{F}_{n,t+k}) - E(X_{nt}|\mathcal{F}_{n,t+k-1})\}$ is uniformly integrable in the Cesàro sense by applying Theorem 3 of Chandra(1989). Fix j, and let

$$Y_{nj} = \sum_{t=1}^{k_n}[E(X_{nt}|\mathcal{F}_{n,t+j}) - E(X_{nt}|\mathcal{F}_{n,t+j-1})].$$

The sequence $\{Y_{nj}, \mathcal{F}_{n,n+j}\}_{n=1}^{\infty}$ is a martingale, and by condition (c) the array $\{[E(X_{nt}|\mathcal{F}_{n,t+j}) - E(X_{nt}|\mathcal{F}_{n,t+j-1})]/c_{nt}, \mathcal{F}_{n,t+j}\}$ is uniformly integrable in the Cesàro sense, and $Y_{nj} \to 0$ in L_1 by Lemma 1.

For $M \ge 1$,

$$\sum_{j=1-M}^{M-1} Y_{nj} = \sum_{t=1}^{k_n} E(X_{nt}|\mathcal{F}_{n,t+M-1}) - \sum_{t=1}^{k_n} E(X_{nt}|\mathcal{F}_{n,t-M}),$$

and hence

$$\sum_{t=1}^{k_n} X_{nt} = \sum_{j=1-M}^{M-1} Y_{nj} + \sum_{t=1}^{k_n}[X_{nt} - E(X_{nt}|\mathcal{F}_{n,t+M-1})] + \sum_{t=1}^{k_n} E(X_{nt}|\mathcal{F}_{n,t-M}).$$

The triangle inequality and the L_1-mixingale property give

$$E\Big|\sum_{t=1}^{k_n} X_{nt}\Big| \le \sum_{j=1-M}^{M} E\Big|Y_{nj}\Big| + \sum_{t=1}^{k_n} E\Big|X_{nt} - E(X_{nt}|\mathcal{F}_{n,t-M-1})\Big| + \sum_{t=1}^{k_n} E\Big|E(X_{nt}|\mathcal{F}_{n,t-M})\Big|.$$

The same argument as Theorem 1 of Davidson(1993) yields that above terms goes to zero which completes the proof.

Proof of Theorem 2. We note that for each $n \ge 2$, $\epsilon > 0$, $P\{|\sum_{t=1}^{k_n} X_{nt} - \sum_{t=1}^{k_n} X'_{nt}| > \epsilon\} = P\{\bigcup_{t=1}^{k_n}\{X_{nt} \neq X'_{nt}\}\} \le \sum_{t=1}^{k_n} P\{|X_{nt}| > c_{nt}k_n\} = \frac{1}{k_n}\sum_{i=1}^{k_n} k_n P\{|X_{nt}/c_{nt}| > k_n\}$, so that the condition (c) entails $\sum_{t=1}^{k_n} X_{nt} - \sum_{t=1}^{k_n} X'_{nt} \to 0$ in probability. Thus to prove the theorem it suffices to verify that $\sum_{t=1}^{k_n} X'_{nt} \to 0$ in probability.

Fix j, and let $Y'_{nj} = \sum_{t=1}^{k_n}[E(X'_{nt}|\mathcal{F}_{n,i+j}) - E(X'_{ni}|\mathcal{F}_{n,t+j-1})]$. We first show that $E(Y_{nj})^2 \to 0$ as $n \to \infty$, and hence $Y'_{nj} \to 0$ in probability. Since $E(X'_{nt}|\mathcal{F}_{n,t+j}) - E(X'_{nt}|\mathcal{F}_{n,t+j-1})$, $1 \le i \le k_n$, form a martingale difference sequence and hence are orthogonal elements of L^2 and $E((E(X'_{nt}|\mathcal{F}_{n,t+j}) - E(X'_{nt}|\mathcal{F}_{n,t+j-1}))^2 \le E(X'_{nt})^2$, we have

$$\begin{aligned}
E(Y'_{nj})^2 &= E\Big(\sum_{t=1}^{k_n}[E(X'_{nt}|\mathcal{F}_{n,t+j}) - E(X'_{nt}|\mathcal{F}_{n,t+j-1})]\Big)^2 \\
&\le \sum_{t=1}^{k_n} E(X'_{nt})^2 \\
&= \sum_{t=1}^{k_n}\sum_{j=1}^{k_n}\int_{\{j-1<|X_{nt}/c_{nt}|\le j\}} X_{nt}^2\,dP \\
&\le \sum_{t=1}^{k_n} c_{nt}^2 \sum_{j=1}^{k_n} j^2\Big(P\{|X_{nt}/c_{nt}| > j-1\} - P\{|X_{nt}/c_{nt}| > j\}\Big) \\
&= \sum_{t=1}^{k_n} c_{nt}^2\Big[P\{|X_{nt}/c_{nt}| > 0\} - k_n^2 P\{|X_{nt}/c_{nt}| > k_n\} \\
&\qquad + \sum_{j=1}^{k_n-1}((j+1)^2 - j)P\{|X_{nt}/c_{nt}| > j\}\Big]
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{t=1}^{k_n} c_{nt}^2 + c_1 \sum_{j=1}^{k_n} \Big[((j+1)^2 - j^2) \sum_{t=1}^{k_n} P\{|X_{nt}/c_{nt}| > j\} c_{nt}^2 \Big] \\
&\leq \sum_{t=1}^{k_n} c_{nt}^2 + c_2 \sum_{j=1}^{k_n} \sum_{t=1}^{k_n} j P\{|X_{nt}/c_{nt}| > j\} c_{nt}^2 \\
&\leq \sum_{t=1}^{k_n} c_{nt}^2 + c_3 \frac{1}{k_n^2} \sum_{j=1}^{k_n} \sum_{t=1}^{k_n} j P\{|X_{nt}/c_{nt}| > j\},
\end{aligned}$$

where c_1, c_2 and c_3 are unimportant positive constants, the third equality comes from Lemma 5.1.1(4) of Chow and Teicher(1988) and the last inequality comes from (a). By (b), the first term above goes to zero as $n \to \infty$ and by (c) the second term goes to zero.

Now we can write as in the Theorem 1,

$$\sum_{t=1}^{k_n} X'_{nt} = \sum_{j=1-M}^{M-1} Y'_{nj} + \sum_{t=1}^{k_n} [X'_{nt} - E(X'_{nt}|\mathcal{F}_{n,t+M-1})] + \sum_{t=1}^{k_n} E(X'_{nt}|\mathcal{F}_{n,t-M}).$$

Then

$$\begin{aligned}
&P\Big\{\Big|\sum_{t=1}^{k_n} X'_{nt}\Big| > \epsilon\Big\} \\
&\leq P\Big\{\Big|\sum_{j=1-M}^{M} Y'_{nj}\Big| > \epsilon/3\Big\} + P\Big\{\Big|\sum_{t=1}^{k_n} [X'_{nt} - E(X'_{nt}|\mathcal{F}_{n,t+M-1})]\Big| > \epsilon/3\Big\} \\
&\quad + P\Big\{\Big|\sum_{t=1}^{k_n} E(X'_{nt}|\mathcal{F}_{n,t-M})\Big| > \epsilon/3\Big\} \\
&\leq \sum_{j=1-M}^{M} P\Big\{|Y'_{nj}| > \frac{\epsilon}{6M}\Big\} + \frac{3}{\epsilon} \sum_{t=1}^{k_n} E|X'_{nt} - E(X_{nt}|\mathcal{F}_{n,t-M-1})| \\
&\quad + \frac{3}{\epsilon} \sum_{t=1}^{k_n} E|E(X'_{nt}|\mathcal{F}_{n,t-M})|.
\end{aligned}$$

Now by similar argument as in Theorem 1 of Davidson(1993), we can show that $\sum_{t=1}^{k_n} X'_{nt} \to 0$ in probability, which completes the proof of theorem.

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