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〈연구논문〉

# Quantum Ballistic Transport in a Two-Dimensional Electron Gas

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# 2차원 전자개스에서 양자 탄동적 수송 현상

**최점수 · 정문성**

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**Abstract** — A hyperbolic model is used to understand quantum ballistic transport through a microscopic constriction in a two-dimensional electron gas. The constriction is given by two hyperbolas of $\beta=\beta_o$ and $\pi-\beta_o$ in elliptic coordinates $(\alpha, \beta)$. The quantized conductance $G$ of the constriction is calculated by using even Mathieu's functions $Se_n(s, \beta)$ which satisfy Schrödinger's equation and the hyperbolic boundary conditions. It is found that the number of channels, $N_c$, depends not only on the constriction width $W$ but also on the curvature-related coordinate $\beta_o$. It is also found that tunneling, depending on $\beta_o$, is the important factor to determine the shape of the quantized $G$-curve. At a fixed $W$, $G$ changes continuously with $\beta_o$ but sometimes undergoes a discrete change when $N_c$ changes by one due to large change in $\beta_o$. For large $\beta_o$, where tunneling is impossible, there exist only a discrete change in $G$.

**요 약** — 쌍곡선 모델을 사용하여 미시 통로점을 통과하는 2차원 전자들의 양자 탄동적 수송현상을 연구하였다. 통로점은 타원좌표계$(\alpha, \beta)$에서 $\beta=\beta_o$, $\pi-\beta_o$로 주어지는 두 쌍곡선으로 기술하였다. 양자화된 88컨덕턴스 $G$는 타원좌표계에서 주어진 슈뢰딩거 방정식과 쌍곡선 경계조건을 만족하는 짝 매튜 함수를 이용하여 계산하였다. 그 결과는 채널수 $N_c$는 통로점 폭 $W$ 뿐만 아니라 곡률 관련좌표 $\beta_o$에 의존함을 나타내었다. 또한 곡률에 의존하는 터널링도 양자화된 $G$의 그래프의 모양을 나타내는 중요한 요소임을 나타내 주었다. 고정된 통로폭에서 $N_c$가 일정한 $\beta_o$ 영역에서는 $\beta_o$의 연속적 변화에 $G$는 연속적으로 변화하였지만 $\beta_o$가 크게 변화할 때는 $N_c$가 변화하여 $G$는 불연속적으로 변화하였다. 만일 터널링이 거의 허용이 안되는 $\beta_o$의 영역에서는 $G$는 계단식의 변화만 보여주었다.

## 1. Introduction

In the current work we study the shape effects of the quantized conductance in a ballistic constriction. When the width, $W$, of the constriction is formed in the two-dimensional electron gas (2DEG), the conductance $G$ has been observed to increase stepwisely [1-5]. The observed $G$ is explained by the Landauer-Büttiker formula [6-8]

$$G=(2\frac{e^2}{h})\,\Sigma T_n, \tag{1}$$

where $T_n$ is the transmission coefficient for the transverse state $n$. When $T_n=1$, Eq. (1) reduces to the empirical relation $G=(2e^2/h)\,N_c$, where $N_c$ is the number of channels available for a ballistic transport [9]. The $N_C$ depends on the width $W$ of the constriction and is semiclassically given to be the greatest integer not more than $k_F W/\pi$. In the two parallel boundaries, there exist the quantized transverse energy $\lambda_m$. The $\lambda_n$ becomes high as the width $W$ becomes small. Thus, only the difference $E_F - \lambda_n$ is available for motion along the longitudinal direction and then $\lambda_n$ behaves like an effective potential barrier for transport through the constriction. The dominant factor in determining $\lambda_m$ is the shape of the constriction at the bottle-neck. It is known that the existence of a long channel with two quasi-parallel boundaries is not essential condition for quantized energy level $\lambda_m$. If $\lambda_m$ is given, we can evaluate the transmission coefficient $T_n$ and obtain the quantized conductance G as a function of shape.

For theoretical calculations of $T_n$, several models of constrictions have been used. They are wide-narrow-wide waveguide [10], plane-channel-plane [11], parabola [12], hyperbola [13-15], delta-function [16], and arbitrary taper model [17]. Among those models, the hyperbolic constriction is of much interest because it is a coordinate shape. By the way, the saddle-point potential model [18] is, even though it is not realistic, good in explaining the quantization of conductance.

It is argued [19] that the conductance quantization phenomena is quite fragile, and is easily destroyed by decreasing the elastic and the inelastic scattering lengths compared with the constriction size. As the length of the constriction is increased, the conductance quantization effect smoothly goes over to the universal quantization regime since the system changes from being ballistic to diffusive.

In the present work, we first use the hyperbolic model to understand the quantum ballistic transport through the constriction and examine the shape effects in quantization of the conductance $G$. To do that, we solve the two-dimensional Schrödinger's equation to obtain the transmission coefficient $T_n$ for a hyperbolic constriction. Since the hyperbola is a coordinate shape, we can find the conductance $G$ as a function of shape factors, $\beta_o$ and $W$. This will be numerically made. In the current work following the previous work [15, 21], we will focus on the curvature-dependence of the channel number. The experimental work on that subject has not been made yet.

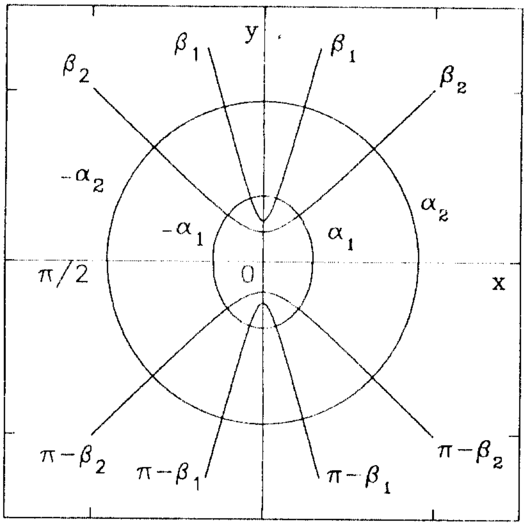


**Fig. 1.** Elliptic Coordinates ($\alpha$, $\beta$). The transformation is $x = c \sinh\alpha \sin\beta$ and $y = c \cosh\alpha \cos\beta$, where $- < \infty < \alpha$, $0 \le \beta \le \pi$, and $c$ is the scale factor. The constriction consists of two hyperbolas of $\beta = \beta_o$ and $\pi - \beta_o$. The angular coordinate $\beta$ represents the angle between the $\psi$-axis and the coordinate. Thus the difference $\pi/2 - \beta_o$ refers the slope of the hyperbola. Here, $\beta_1 = 15°$ and $\beta_2 = 45°$.

## 2. Hyperbolic Model

We consider a ballistic point contact connecting two wide regions of 2DEG. The point contact can be regarded as a narrow channel which has a hyperbolic shape. By convenience, we introduce elliptic coordinates ($\alpha$, $\beta$) defined by $x = c \sinh\alpha \sin\beta$ and $y = c \cosh\alpha \cos\beta$, where $-\infty \le \alpha \le \infty$, $0 \le \beta \le \pi$ and $c$ is the scale factor (see Fig. 1). The constriction consists of two hyperbolas of $\beta = \beta_o$ and $\beta = \pi - \beta_o$. The diference $\pi/2 - \beta_o$ refers to the slope of the hyperbola of $\beta_o$. The width, $W$, of the constriction at the bottle-neck ($\alpha=0$) is $2c\cos\beta_o$.

The two-dimensional Schrödinger equation for an electron in the potential $V(r)$ is written in the form

$$-\frac{h^2}{2m}\frac{1}{c^2(\cosh^2\alpha-\cos^2\beta)} \times\left(\frac{\partial^2\Psi}{\partial\alpha^2}+\frac{\partial^2\Psi}{\partial\beta^2}\right)+V\Psi=E\Psi \tag{2}$$

It is assumed that $V=0$ for $\beta_o<\beta<\pi-\beta_o$ and $V=\infty$ for $\beta<\beta_o$ and $\beta>\pi-\beta_o$. Thus, putting $\Psi(\alpha,\beta)=\psi(\alpha)\chi(\beta)$, Eq. (2) leads to the two one-dimensional equations

$$\frac{d^2\chi}{d\beta^2}+(b-s\cos^2\beta)\,\chi=0, \tag{3}$$

$$\frac{d^2\psi}{d\alpha^2}-(b-s\cosh^2\alpha)\,\psi=0, \tag{4}$$

where

$$s=2mc^2E/\hbar^2, \tag{5}$$

$$b=2mc^2\lambda/\hbar^2. \tag{6}$$

Here, $\lambda$ is the constant of separation, which means an effective potential barrier height in Eq. (4). The constants $s$ and $b$ are dimensionless energies associated with $E$ and $\lambda$. Equations (3) and (4) are called Mathieu's equation and Mathieu's modified equation, respectively [20].

The solution of Eq. (3) with the hard wall boundary conditions are even Mathieu's functions $Se_n(s,\beta)$, where n is the number of zeros in the interval $0\leq\beta\leq\pi$. For a given constriction of $\beta_o$ such that $Se_n=0$ at $\beta=\beta_o$, we can obtain characteristic value $b_n$ for every $s$. The $b_n$ or $\lambda_n$ is the quantized transverse energy obtained from the angular equation, Eq. (3). Since the difference $E_F-\lambda_n$ is the energy available for the longitudinal transport through the constriction, $\lambda_n$ plays the role as an effective potential. Thus, if the energy lies between $\lambda_n$ and $E_F$, then the corresponding state contribute to the conduction and is regarded as a

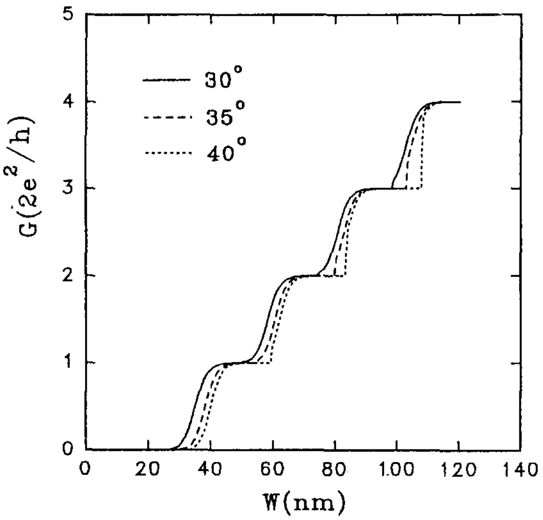


**Fig. 2.** Conductance $G$ of the Hyperbolic Constrictions. Here, $\beta_o=30$, 35 and 40°. The $G$ are calculated by solving Schrödinger's equation in elliptic coordinates. Three constrictions have the same eigenmodes in the region $0<W<120$ nm (see Table 1). The shape effect due to the different slope of the constriction. is the slope of the $G$-curve.

channel. Another contribution to the conduction is the tunneling which depends on the potential geometry. Of course, this is also determined by the constriction geometry.

The major contribution to the potential is due to the geometry of the bottle-neck. Then we expand $\cosh^2\alpha$ about $\alpha=0$ and keep the first two terms. Thus Eq. (4) is approximately reduced to

$$\frac{d^2\psi}{c^2d\alpha^2}+\frac{2m}{\hbar^2}\left[E_F-(\lambda_n-E_F\alpha^2)\right]\psi=0, \tag{7}$$

where $\lambda_n=\hbar^2b_n/(2mc^2)$. Equation (7) is Schrödinger's equation for a scattering problem for a potential $\lambda_n-E_F\alpha^2$. For a such type of potential, the transmission coefficient $T_n$ is given by [22]

$$T_n=\frac{1}{1+\exp(-\pi\,\varepsilon_n)}, \tag{8}$$

$$\varepsilon_n=(s-b_n(s))/s^{1/2}. \tag{9}$$

Substitution of Eq. (8) into Eq. (1) yields the conductance $G$ of a hyperbolic constriction as a func-

**Table 1.** First Few modes $Se_n(s, \beta)$ and Number of Channels $N_c$. For $s<150$, stable even Mathieu functions are limited by the boundary condition, $Se_n(s, \beta)$ at $\beta=\beta_o$ and $\pi-\beta_o$. The constriction is given by two hyperbolas of $\beta=\beta_o$ and $\pi-\beta_o$. The $N_c$ at $W=90$ nm is the number of channels, *i.e.* even Mathieu's functions denoted by asterisk*.

| Constriction | First four stable modes | $N_c$ at $W=90$ nm |
|---|---|---|
| $\beta_o=15°$ | $Se_6$*, $Se_7$, $Se_8$, $Se_9$ | 1 |
| 20° | $Se_5$*, $Se_6$*, $Se_7$, $Se_8$ | 2 |
| 22.5° | $Se_4$*, $Se_5$*, $Se_6$, $Se_7$ | 2 |
| 30° | $Se_3$*, $Se_4$*, $Se_5$*, $Se_6$ | 3 |
| 35° | $Se_3$*, $Se_4$*, $Se_5$*, $Se_6$ | 3 |
| 40° | $Se_3$*, $Se_4$*, $Se_5$*, $Se_6$ | 3 |
| 45° | $Se_2$*, $Se_3$*, $Se_4$*, $Se_5$ | 4 |

tion of $\beta_o$ and s (or the constriction width $W$).

## 3. Calculations and Results

The solution which satisfies Eq. (3) and the hard wall boundary condition is only even Mathieu's functions $Se_n(s, \beta)$ with $n\geq 2$. [15,20] With the table of the expansion coefficients of $Se_n(s, \beta)$ [20], we made numerical calculations of $Se_n(s, \beta)$ for $0\leq s\leq 100$ and $0\leq\beta\leq 70°$. For $s>100$, we use approximate expression of $Se_n(s, \beta)$ [20]. From the obtained $Se_n(s, \beta)$ we found s such that $Se_n=0$ at $\beta=\beta_o$ and $\pi-\beta_o$. We also used the numerical table to obtain the characteristic values $b_n(s)$ for $a$ given $s$. Among all obtained $b_n(s)$, we chose the appropriate $b_n(s)$ under the condition of stable solution for $Se_n(s, \beta)$. The stable mode $n$ can contribute to the transport through the constriction if $s$ is less than $(0.5k_FW/\cos\beta_o)^2$, where $W=2\sqrt{s}\ \cos\beta_o/k_F$ *and* $k_F\simeq 0.15$/nm. Thus the channel number $N_c$ is limited.

Using Eqs. (8) and (9) along with the obtained $b_n(s)$, we calculated the transmission coefficient $T_n$ and then the conductance G. These results are shown in Figs. 2 and 3. In Fig. 2, we plot the quantized G for hyperbolic constrictions of $\beta_o=$ 30, 35, 40°. These choice of $\beta_o$ are made because such constrictions have the same functions $Se_3$, $Se_4$, $Se_5$, $Se_6$, $\cdots$ as the first few possible modes (see Table 1). In the region of interest for $W$, they have the same $N_c$. Thus, we can know the shape effects in $G$-curve for fixed $G$. According to Fig. 2, the large-$\beta_o$ curves are clearly step-like and the small-$\beta_o$.

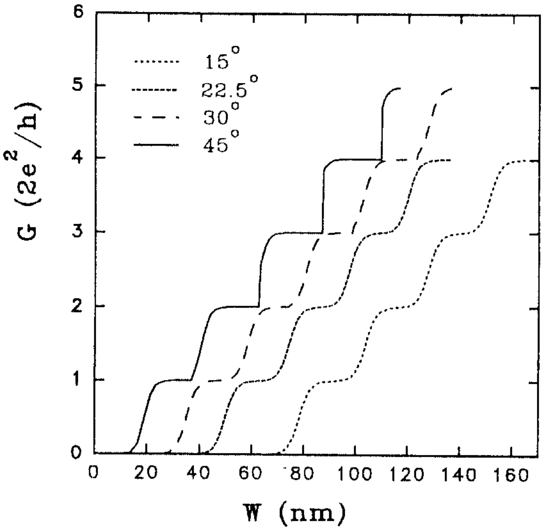


**Fig. 3.** Conductance $G$ of the Hyperbolic Constrictions. Here, $\beta_o=15$, 22.5, 30 and 45°. The G are calculated by solving Schrödinger's equation in elliptic coordinates. These graphs show the shape effect in both $N_c$ and the slope of $G$-curves. At the same $W$, $N_c$ depends on $\beta_o$ only.

The feature mentioned above should be modified if we expand the values of $\beta_o$ beyond the region of the same $N_c$. In Fig. 3, we have shown the quantized $G$ for constrictions of $\beta_o=15$, 22.5, 30, 45°. Differently from Fig. 2, $N_c$ in Fig. 3 changes with $\beta_o$ even when $W$ is fixed. For constrictions of $\beta_o=45°$, there appears the first mode $Se_2(s, \beta)$ in addition to the modes for the constriction of $\beta_o=40°$(see Table 1). According to Fig. 3 and Table 1, the constriction of $\beta_o=45°$ has 1 more channel than that of $\beta_o=30°$ and 3 more channels than that of $\beta_o=15°$. This implies that quantization of transverse states takes place more easily for a smoothly changing shape of the constriction (*i.e.* for large $\beta_o$) than for a sharply changing shape (*i.e.* for small $\beta_o$). The difference $G$-curves of same $N_c$ are due to different tunneling over constrictions of $\beta_o=15°$ through 45°.

Tunneling effect is smoothing the $G$-curve which becomes slant-like. It is noted that the current work have shown the curvature-dependence of $N_c$.

We also obtained the quantized $G$ for constrictions of $\beta_o > 45°$. In the region of $\beta_o > 45°$, we have no additional channels. As mentioned before, $Se_2(s, \beta_o)$ is the lowest mode solution among the even Mathieu's functions $Se_2(s, \beta_o)$. Thus, we can not expect a new channel as obtained in Fig. 3 and Table 1. We have also found that for $\beta_o < 10°$, no quantized channel available for transport for $W < 120$ nm. This implies that the associated confining region is a little smaller than needed for the 'usual' quantization. By the usual quantization, we mean the energy quantization in the hard-wall boundaries. The fact that for $\beta_o < 10°$, there exist no quantized transverse states does not imply no transport through the constriction. The reason is that for large $W$, transport should be possible irrespective of the slope of constriction. Since there are no quantized barrier, the conductance may be proportional to the width $W$.

According to the results for $\beta_o \leq 45°$, we argue that a sort of smoothness and length of the channel are necessary for considerable number of quantized steps of $G$. By the results for $\beta_o \geq 45°$, we also argue that the strict smoothness is not needed for quantization. This conclusion is consistent with experimental and theoretical works [1, 2, 13, 14]. Kawabata [13] argued that the existence of a long quasi-one-dimensional channel with parallel boundaries is not an essential condition for the quantization of conductance. Yosefin and Kaveh [14] argued that conductance quantization can be obtained in a quite general smooth confining potential, provided that it creates a narrow bottleneck region.

## 4. Conclusions

In the present work, we have reported several features in quantum ballistic transport in the constriction. First, a sort of smoothness and length of the channel are necessary to obtain a set of quantized steps of the conductance. However, such a kind of requirement is not shown to be strict. Second, the number of channels $N_c$ is a function of the constriction width and curvature. The experimental attempts to show the curvature dependence of $N_c$ has not been made yet. Third, tunneling is a major factor to determine the shape of the G-curve. As a constriction becomes smooth and long, the G-curve becomes stair-like because of less tunneling.

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