비압축성 물체의 압력해 안정화를 위한 압력연속여분치의 매개변수 연구

Parametric Study on the Pressure Continuity Residual for the Stabilization of Pressure in Incompressible Materials

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요 약

비압축성 물체를 위한 일반적인 유한요소 공식화는 흔히 사용되는 사각형요소에서조차 압력해의 진동화 (oscillations) 또는 pressure modes 현상을 나타낸다. 압력해의 안정화를 위한 규준은 소위 Babuška-Brezzi 안정조건이며, 위의 요소들은 이 조건을 만족시키지 못한다. 본 연구에서는 선형변위해와 상수값의 압력해를 갖는 사각형요소 사용시 압력해를 안정화시키기 위해 요소의 변에서 발생하는 불연속압에 근거한 압력연속여분치를 사용한다. 이 압력여분치를 비압축성 탄성론으로부터 유도되는 Q1P0요소에 적용하며 매개변수의 변화에 따른 수치해의 안정화의 정도를 연구한다. 압력해는 압력 여분치 사용시 안정화될 수 있으며, 해의 안정화는 매개변수에 민감성을 나타내었다.

Abstract

The conventional finite element formulations for incompressible materials show pressure oscillations or pressure modes in four-node quadrilateral elements of commonly used displacement and pressure interpolations. The criterion for the stability in the pressure solution is the so-called Babuška-Brezzi stability condition, and the above elements do not satisfy this condition. In this study, a pressure continuity residual based on the pressure discontinuity at element interfaces is used to study the stabilization of pressure solutions in bilinear displacement-constant pressure four-node quadrilateral elements. This pressure residual is implemented in Q1P0 element derived from the conventional incompressible elasticity. The pressure solutions can be stable with the pressure residual though they exhibit sensitivity to the stabilization parmaeters. Parametric study for the solution stabilization is also discussed.

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1. INTRODUCTION

Pressure solutions in conventional finite element formulations using natural combination of displacement (velocity in fluid problems) and pressure interpolation functions exhibit pressure oscillations or pressure modes in incompressible or nearly incompressible materials. During the past several decades, many researchers have studied this topic since Taylor and Hood(1973) discovered difficulty in the pressure solution.

The use of equal-order interpolation on conforming quadrilateral elements, wherein the same interpolation functions are used to represent velocity and pressure, caused difficulty in the pressure solution. Even when several types of mixed interpolations were employed, however, there were cases where numerical dif ficulties were encountered.

The simplest elements, which employ linear or bilinear interpolation functions for velocity and constant approximation for pressure, have been found to work well in some cases and poorly in others. For certain combinations of boundary conditions and element distributions over a domain, the sloutions display acceptable velocities but totally spurious pressures, severe oscillations often called a checkerboard pressure mode. The best known example which suffers from a checkerboard pressure mode on rectangular grids, and erroneous pressure modes and velocity inaccuracy on more general quadrilateral grids, can be found in Sani et al. (1981). Similar behavior was also observed when higher-order elements were used.

The mathematical framework for understanding the behavior of mixed method for the Stokes problem was provided by Babuška (1971) and Brezzi(1974). The key requirement boiled down to satisfaction of a stability condition which involves both velocity and pressure spaces. This is the so-called Babuška-Brezzi condition.

In the 1980's, many new ideas and developments have been made for the stabilization of pressure solution and the stability criterion, for example, Oden et.al.(1982), Pitkä ranta and Stenberg(1984), Pitkäranta and Saarinen (1985), Hughes et.al.(1986). In 1987, Hughes and Franca suggested a generalized formulation which involves the residuals of pressure continuity and of the equilibrium equations. This formulation circumvents the Babuška-Brezzi condition and makes it possible to use a natural combination of displacement and pressure interpolants. However, in the use of this formulation, the relationship between the accuracy of displacement and pressure solutions and the stability parameter needs detailed study.

In this paper, a pressure continuity residual based on the pressure discontinuity at element edges is used to stabilize pressure solutions in the formulation of bilinear displacement-constant pressure four-node quadrilateral element (Q1P0). The stability of the pressure solutions is studied with various stability parameters. The accuracy of the displacement and pressure solutions are compared with those of other conventional elements. The sensitivity of the pressure solutions to the stabilization parameters is also discussed.

2. INCOMPRESSIBLE ELASTICITY FORMULATION FOR THE STABILIZATION OF PRESSURE

Let Ω be an open bounded region in $R^{n_{sd}}$, where n_{sd} is the number of space dimensions ($n_{sd}=2$ or 3), with piecewise smooth boundary Γ. The standard displacement-pressure formulation of isotropic incompressible elasticity is:

$$\mathrm{div}\ \sigma + b = 0 \quad \text{in } \Omega \qquad (1)$$
$$\mathrm{div}\ u = 0 \quad \text{in } \Omega \qquad (2)$$
$$\sigma = 2\mu\varepsilon - pI \quad \text{in } \Omega \qquad (3)$$
$$u = u^* \quad \text{on } \Gamma_u \qquad (4)$$
$$\sigma \cdot n = t^* \quad \text{on } \Gamma_{td} \qquad (5)$$

Here σ is the Cauchy stress tensor, b is body force, u is displacement (or velocity in fluid mechanics), μ is shear modulus (or viscosity in fluid mechanics), p is pressure, I is the identity tensor, and ε is the symmetric part of the displacement gradient. Equation (2) gives the incompressibility condition. The boundary Γ consists of two subregions, Γ_u and Γ_t, which are the displacement and traction boundaries, respectively, and

$$\Gamma = \Gamma_u \cup \Gamma_t \qquad (6)$$
$$\phi = \Gamma_u \cap \Gamma_t \qquad (7)$$

The unit outward normal vector to Γ is denoted by n.

Let $\mathcal{U}$ and $\mathcal{V}$ be the spaces of displacement trial solutions and test functions, and $\mathcal{P}$ be the space of pressures. Let u and δu denote displacement trial solutions and test functions, and p and δp denote pressure trial solutions and test functions, respectively. The conventional weak formulation for isotropic incompressible materials is

$$\int_\Omega \delta\varepsilon^T \sigma d\Omega - \int_\Omega \delta p^T (\mathrm{div}\ u) d\Omega = \int_\Omega \delta u^T b d\Omega + \int_{\Gamma_t} \delta u^T t^* d\Gamma \qquad (8)$$

Equation (8) can be rewritten as follows by using equation (3)

$$\int_\Omega \delta\varepsilon^T 2\mu\varepsilon d\Omega - \int_\Omega \delta(\mathrm{div}\ u)^T p d\Omega - \int_\Omega \delta p^T (\mathrm{div}\ u) d\Omega = \int_\Omega \delta^T b d\Omega + \int_{\Gamma_t} \delta u^T t^* d\Gamma \qquad (9)$$

where the trial solutions u and p, and the test functions δu and δp are defined as follows:

$$u \in \mathcal{U};\ \mathcal{U} = \{u \mid u \in C^0,\ u = u^* \text{ on } \Gamma_u\} \qquad (10)$$
$$\delta u \in \mathcal{V};\ \mathcal{V} = \{\delta u \mid \delta u \in C^0,\ \delta u = 0 \text{ on } \Gamma_u\} \qquad (11)$$
$$p \in \mathcal{P},\ \delta p \in \mathcal{P};\ \mathcal{P} = \{p \mid p \in C^{-1}\} \qquad (12)$$

The above weak formulation is not stable for pressure solutions unless specific displacement and pressure interpolations are chosen. In particular, pressure oscillations, or pressure modes (often called checkerboarding which are caused by singularities in the global equations) occur in the bilinear displacement-constant pressure four-node quadrilateral element(Q1P0).

Hughes and Franca(1987) modified the weak formulation (9) as follows to stabilize the pressure modes

$$\int_\Omega \delta\varepsilon^T 2\mu\varepsilon d\Omega - \int_\Omega \delta(\mathrm{div}\ u)^T p d\Omega - \int_\Omega \delta p^T (\mathrm{div}\ u) d\Omega - \int_{\tilde\Omega} \frac{\alpha l^2}{2\mu} \delta(\mathrm{div}\ \sigma)^T (\mathrm{div}\ \sigma) d\Omega - \int_{\tilde\Gamma} \frac{\beta l}{2\mu} [[\delta p]]^T [[p]] d\Gamma = \int_{\tilde\Omega} (\delta u^T + \frac{\alpha l^2}{2\mu} (\mathrm{div}\ \sigma)^T) b d\Omega + \int_{\Gamma} \delta u^T t^* d\Gamma \qquad (13)$$

where α and β are nondimensional stabilization parameters ($\alpha \geq 0$ and $\beta \geq 0$), l is the mesh parameter, and $[[\cdot]]$ is the jump operator. The domain $\tilde\Omega$ denotes element interiors, and $\tilde\Gamma$ consists of the element interfaces so $\tilde\Gamma = \bigcup_e \Gamma_e - \Gamma$, where e refers to an element. the above weak formulation involves the addition of 'least-squares' forms of the following residuals: one is the equilibrium equation (or momentum equation) residual, the other is the pressure continuity residual on element interfaces. Note that, in low order element such as Q1P0, div σ almost vanishes and only the pressure continuity residual is left. These terms render the formulation to be coercive, in contrast to the classic Galerkin formulation, and enable the Babuška-Brezzi condition to be avoided. So this formulation can provide stable pressure solutions for seemingly arbitrary combinations of dispalcement and pressure interpolations. In the above formulation, however, a careful choice of the parameters is required to prevent a loss of accuracy in the solution. The deficiency of this formulation is that the global nature of the pressure residual terms makes the method awkward to implement into general finite element codes.

Silvester and Kechkar(1990) suggested a local jump formulation wherein jump terms are calculated at the interior interfaces for 2×2 groups of elements, which is termed a macro element. This method makes the implementation more straightforward.

In the study, an elemental pressure discontinuity operator is introduced into the pressure continuity residual for the stabilization of pressure. The global pressure residual matrix can be constructed by the assembly of the pressure residual in each element.

The weak formulation which will be used in the finite element formulation of the Q1P0 element is

$$\int_{\Omega}[\delta\varepsilon^T 2\mu\varepsilon-\delta(\text{div } u)^T p-\delta p^T(\text{div } u)]d\Omega$$
$$-\alpha_1\sum_{e=1}^{N_{el}}\int_{\Gamma_e}[[\delta p]]_e^T[[p]]_e d\Gamma$$
$$=\int_{\Omega}\delta u^T b d\Omega+\int_{\Gamma_t}\delta u^T t^* d\Gamma \qquad (14)$$

where α_1 is a stability parameter, N_{el} is the total number of elements, Γ_e denotes element interface, $[[\cdot]]_e$ is the elemental pressure discontinuity operator.

3. FINITE ELEMENT FORMULATION FOR Q1P0 ELEMENT

Consider the bilinear displacement-constant pressure four-node quadrilateral element, which is one of the most convenient elements is the finite element analysis. We use the following trial functions in each element

$$u = N\,d = \sum_{I=1}^{4} N_I(\xi,\eta)\,d_I \qquad (15)$$

$$\varepsilon = B\,d = \sum_{I=1}^{4} B_I(\xi,\eta)\,d_I \qquad (16)$$

$$\text{div } u = H\,d = \sum_{I=1}^{4} H_I(\xi,\eta)\,d_I \qquad (17)$$

$$p = N^p(\xi,\eta)\,p \qquad (18)$$

$$N_I = \frac{1}{4}(1+\xi\xi_I)(1+\eta\eta_I) \qquad (19)$$

$$B_I = \begin{bmatrix} N_{I,x} & 0 \\ 0 & N_{I,y} \\ N_{I,y} & N_{I,x} \end{bmatrix} \qquad (20)$$

$$H_I = [\,N_{I,x} \quad N_{I,y}\,] \qquad (21)$$

$$N^p = [1] \qquad (22)$$

where N and N^p are the displacement and pressure shape functions, respectively, d is the vector of nodal displacements u_x and u_y, ξ and η are the referential coordinates, $\xi \in [-1, +1]$ and $\eta \in [-1,+1]$. The pressure is assumed to be constant within each element. Substituting (15)-(22) into (14) gives

$$\delta d^T\int_{\Omega}B^T D^{dev} B d\Omega d - \delta d^T\int_{\Omega}H^T N^p d\Omega p$$
$$-\delta p^T\int_{\Omega}(N^p)^T H d\Omega d - \delta p^T R p$$
$$=\delta d^T\int_{\Omega}N^{pT} b d\Omega + \delta d^T\int_{\Gamma_t}N^T t^* d\Gamma \qquad (23)$$

Here, D^{div} is a diagonal matrix whose components are $(2\mu, 2\mu, 2\mu)$ in two dimensions and $(2\mu, 2\mu, 2\mu, \mu, \mu, \mu)$ in three-dimensional problems, and R is the pressure continuity residual matrix.

The discrete global equations obtained from (23) are

$$\begin{bmatrix} K_{dd} & K_{dp} \\ K_{pd} & K_{pp} \end{bmatrix} \begin{Bmatrix} d \\ p \end{Bmatrix} = \begin{Bmatrix} F^{ext} \\ 0 \end{Bmatrix} \tag{24}$$

where

$$K_{dd} = \sum_{e=1}^{Nel} \int_{\Omega_e} B^T D^{dev} B d\Omega \tag{25}$$

$$K_{dp} = -\sum_{e=1}^{Nel} \int_{\Omega_e} H^T N^p \, d\Omega \tag{26}$$

$$K_{pd} = -\sum_{e=1}^{Nel} \int_{\Omega_e} (N_p)^T \, H d\Omega \tag{27}$$

$$K_{pp} = -R = -\sum_{e=1}^{Ne} \int_{\Omega_e} R_e, \tag{28}$$

$$f^{ext} = \sum_{e=1}^{Nel} \left(\int_{\Omega_e} N^T b d\Omega + \int_{\Gamma_t} N^T t^* d\Gamma \right) \tag{29}$$

4. ELEMENTAL PRESSURE DISCONTINUITY OPERATOR AND PRESSURE CONTINUITY RESIDUAL MATRIX

Suppose one side Γ_i of an element edge Γ_e and let's assume this side to be the interface of element A and B. The pressure discontinuity at this side is defined as

$$[[p]]_{\Gamma_i} = p_A - p_B \tag{30}$$

Let the residuals by the pressure discontinuity at the element edges be denoted by $[[\cdot]]_e$ which is called elemental pressure discontinuity operator herein, then the pressure residual at the edges of the element A is defined as

$$[[p_A]]_e = \sum_{i=1}^{nint} (p_A - p_{Ai}) \tag{31}$$

where n_{int} is the number of interface perelement, p_A is the pressure of element A, and p_{Ai} is the pressure of adjacent element which is in contact with the i'th interface of the element A.

The pressure continuity residual matrix R in equation (28) can be constructed by assembling the pressure residual of each element. To calculate the matrix R, consider an example subdomain shown in Fig. 1. This subdomain has nine elements and unknown pressure at each number of element. Each element has sides of length a and b. From the definition (31), the pressure residual at the interfaces of the first element is

$$[[p_1]]_{e=1} = \overset{1-2}{0} + \overset{2-3}{(p_1 - p_2)} + \overset{3-4}{(p_1 - p_4)} + \overset{4-1}{0} = Z_1 \tag{32}$$

Here the matrix Z_1 and the pressure vector p are

$$Z_1 = [2, \ -1, \ 0, \ -1, \ 0, \ 0, \ 0, \ 0, \ 0] \tag{33}$$

$$p = [p_1, \ p_2, \ \cdots \ p_9]^T \tag{34}$$

The global pressure residual is assembled from the residuals at all element edges, so pressure residual

$$= \alpha_1 \sum_{e=1}^{Nel} \int_{\Gamma_e} [[p]]_e d\Gamma = \alpha_1 \bar{Z} \tag{35}$$

The semi-bandwidth of the matrix $\bar{Z}$ is determined by the maximum difference of adjacent elements numbers. The matrix $\bar{Z}$ is symmetric and positive definite.

The following two forms have been used for the pressure residual matrix R in the study:

Type 1. $R = \alpha_1 \bar{Z}$

$$\text{where } \alpha_1 = \frac{\beta l}{2\mu} \tag{36}$$

Type 2. $R = \alpha_1 \bar{Z}^T \bar{Z}$

$$\text{where } \alpha_1 = \frac{\beta l}{2\mu} \tag{37}$$

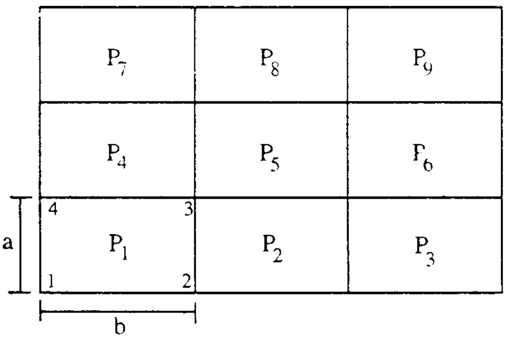


Fig. 1 An example for the pressure continuity residual

5. NUMERICAL EXAMPLES

For the convergence study, the displacement norm (L_2 norm) can be calculated as

displacement norm

$$= [\int_\Omega (u-u^h)(u-u^h)\, d\Omega]^{1/2} \tag{38}$$

and the energy norm (H_1 norm) for compressible or nearly incompressible materials can be calculated as follow:

energy norm

$$= [\frac{1}{2}\int_\Omega (\varepsilon-\varepsilon^h)^T D(\varepsilon-\varepsilon^h)\, d\Omega]^{1/2} \tag{39}$$

For incompressible plane strain, equation (39) can not be used because the stree-strain matrix D is not defined when ν is 0.5. The energy norm can be decomposed into the deviatorin energy norm and the pressure norm as

deviatoric energy norm

$$= [\frac{1}{2}\int_\Omega (\varepsilon-\varepsilon^h)^T D^{dev}(\varepsilon-\varepsilon^h)\, d\Omega]^{1/2} \tag{40}$$

pressure norm

$$= [\int_\Omega (p-p^h)^T (p-p^h)\, d\Omega]^{1/2} \tag{41}$$

In the four-node Q1P0, the pressure is considered to be constant in each element, which makes the magnitude of pressure norm very large compared to that of deviatoric strain energy norm. This makes it difficult to compare (39) and (40)+(41). the following modified form of (39) is also suggested for incompressible materials to consider the both errors in the deviatoric strain energy and pressure:

energy norm*

$$= [\frac{1}{2}\int_\Omega |\varepsilon-\varepsilon^h|^T |\sigma-\sigma^h|\, d\Omega]^{1/2} \tag{42}$$

where the mark 'energy norm*' denotes that the absolute values of the errors in strain and stress are used to evaluate the error of strain energy. In (42), σ^h in each element is calculated by using the constant pressure solution p^h as follow:

$$\sigma^h = 2\mu\varepsilon^h - p^h I. \tag{43}$$

5.1 Convergence Test in Timoshenko Beam Bending Problem

The test problem is a linear, elastic cantilever with a load P at its end as shown in Fig. 2. The solution to this problem can be found in Timoshenko and Goodier (1970).

In the convergence study of this beam problem, incompressible or nearly incompressible plane strain is considered because both volumetric locking and pressure oscillations (or pressure modes) occur in this case. The performance of Q1P0 element with two different types of pressure residuals has been studied by changing the stability parameter β from 0.0001 to 1.0. For various values of β, the y-deflections at point A, are given in Table 1. The y-deflections in Table 1 show that the stiffness matrix becomes more flexible as the stability parameter β increases. Q1P0 element shows significant locking when β is equal to zero.

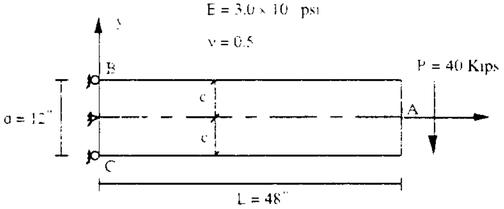


Fig. 2 Linear elastic Timoshenko beam bending problem

Table 1. Deflections in the Timoshenko beam bending problem.

Deflections at point A (u_{FEm}/u_{exact})								
Pressure residual Type 1								
β / ele.	0.0001	0.001	0.005	0.01	0.05	0.1	0.5	1.0
8	0.885	0.886	0.892	0.899	0.947	0.991	1.127	1.173
32	0.964	0.965	0.970	0.976	1.017	1.061	1.266	1.378
128	0.990	0.990	0.993	0.997	1.023	1.051	1.202	1.313
512	0.997	0.997	0.999	1.001	1.017	1.034	1.126	1.199
Pressure residual Type 2								
β / ele.	0.0001	0.001	0.005	0.01	0.05	0.1	0.5	1.0
8	0.885	0.890	0.912	0.936	1.052	1.118	1.228	1.250
32	0.964	0.967	0.976	0.987	1.058	1.123	1.366	1.471
128	0.990	0.991	0.995	1.001	1.033	1.060	1.169	1.244
512	0.997	0.998	1.000	1.003	1.019	1.033	1.085	1.116

In Fig. 3 and 4 the displacement norm and the energy norm of the Q1P0 element are compared with those of other elements; QM6 element by Taylor et al.(1976) and ASQBI element by Belytschko and Bindeman(1991). Poisson's ratio $\nu=0.4999$ in QM6 and ASQBI; whereas $\nu=0.5$ in Q1P0. The stability parameter β is set to be 0.001. In these figures, the displacement norm and the energy norm of Q1P0 element show optimal convergence rates like QM6 or ASQBI though the accuracy Q1P0 is inferior to QM6 or ASQBI (actually the performance of QM6 and ASQBI is outstanding in beam bending problems).

In another approach, prescribed displacement boundary conditions instead of traction boundary conditions has been applied to the beam. The convergence retes of the displacement norm and the energy norm in Q1P0 are still optimal except for large β values (except for β=0.5 or 1.0); whereas the accuracy of the pressure solutions of QM6 and ASQBI becomes worse since they do not satisfy Babuška-Brezzi stability condition. Next example shows this more clearly.

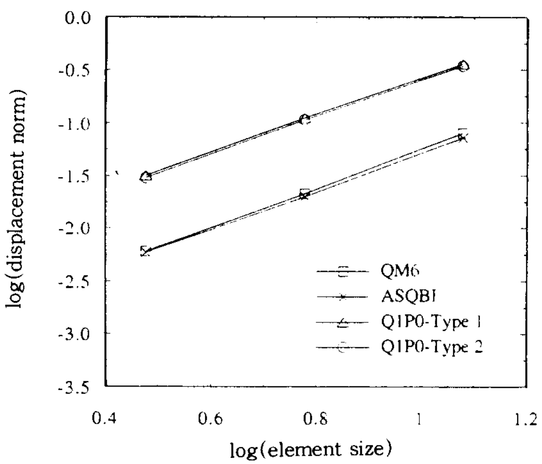


Fig. 3 Convergence rates of displacement norms in several elements(β=0.001 in Q1P0)

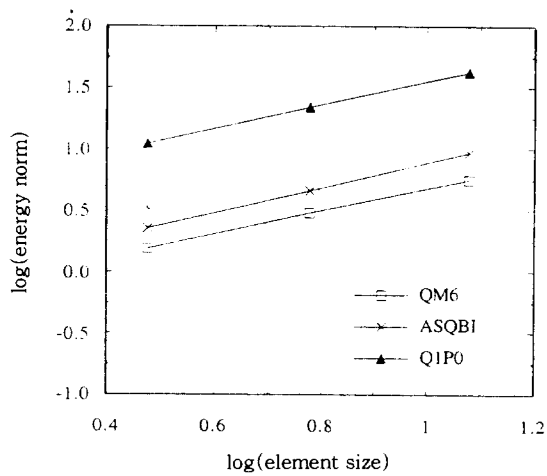
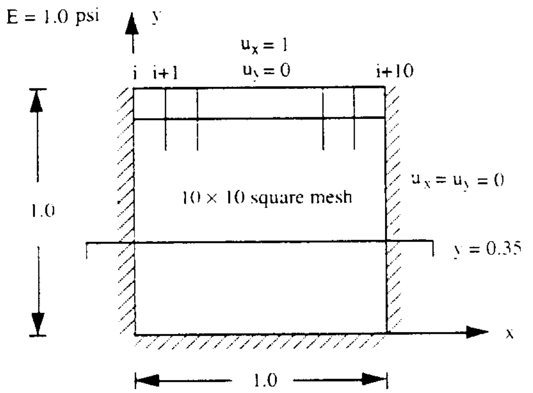


Fig. 4 Convergence rates of energy norms in several elements(β=0.001 in Q1P0)

5.2 Driven Cavity Flow Problem

Another example is the well known driven cavity flow problem. The geometry of the model and the applied boundary conditions are shown in Fig. 5. Two different boundary conditions have been used; one of which, CASE A (often called 'leaky lid' boundary condition), causes pressure oscillations and the other, CASE B (often called 'ramp over one element' boundary condition), causes pressure modes in conventional finite element analyses. In this example, the stability parameter β was fixed to 0.005 and the Type 2 pressure residual has been used.



Boundary conditions

CASE A

$u_x^i = u_x^{i+1} = \cdots = u_x^{i+10} = 1.0$

$u_x = u_y = 0$ elsewhere on Γ

CASE B

$u_x^{i+1} = \cdots = u_x^{i+9} = 1.0$

$u_x = u_y = 0$ elsewhere on Γ

Fig. 5 Driven cavity flow model and two different boundary conditions

The distribution of pressure at y=0.35 for Q1P0, and other elements is shown in Fig. 6, where the boundary condition CASE A has been used. The Smoothed ccurve obtained by post processing, see (Lee et al., 1979), is considered as exact solution. The pressures of QM6 and ASQBI are oscillatory whereas those of Q1P0 with the Type 2 pressur eresidual are stable and accurate. The distribution of pressures in the boundary condition CASE B is shown in Fig. 7. this boundary condition makes the pressure solutions of conventional methods more disastrous, in other words, pressure solutions have positive nealry infinite value and negative value, alternatively which is called pressure modes or checkerboarding. The pressures of QM6 and ASQBI show pressure modes and therfore are not shown in Fig. 7. the pressures of Q1P0 are also stable in this case.

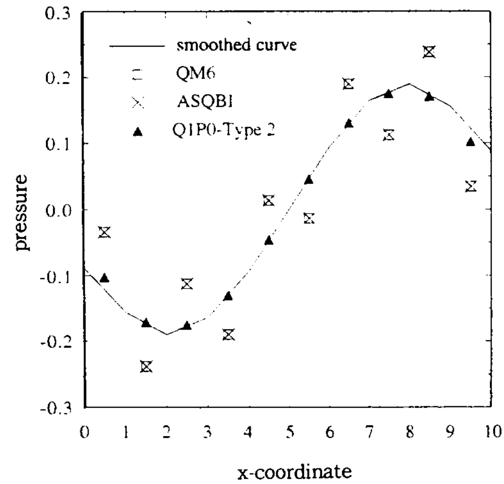


Fig. 6 Pressure distributions at y=0.35 in the driven cavity flow problem with the boundary condition CASE A

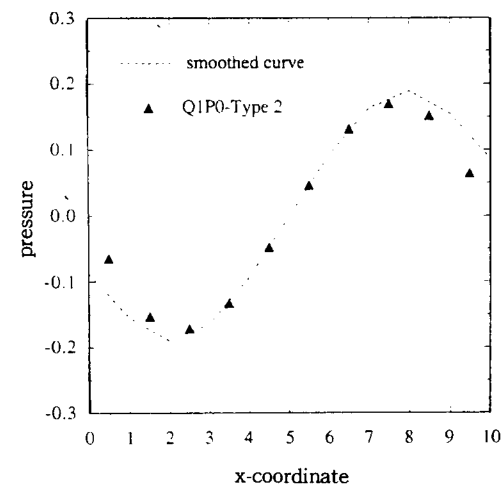


Fig. 7 Pressure distributions at y=0.35 in the driven cavity flow problem with the boudnary condition CASE B (other elements show the pressure modes)

6. CONCLUSIONS

The Q1P0 element derived from the conventional variational principle for incompressible elasticity has been presented for the pressure stabilization of bilinear displacement-constant pressure four-node quadrilaterals. The addition of the pressure continuity residual circumvents the Babuška-Brezzi condition. The stability of the pressure solution was studied as the stability parameter β varied from 0.0001 to 1.0. The displacement and pressure solutions of Q1P0 element with the pressure residual show the following features according to the variation of β.

1) For $0<\beta\leq 0.001$: displacement solution shows a good accuracy. However, pressure oscillations or pressure modes can not be eliminated successfully.

2) For $0.001\leq\beta\leq 0.05$: pressure oscillations or pressure modes are eliminated successfully. Pressure solutions are stable and accurate. Displacement solutions are still accurate.

3) For $0.5\leq\beta\leq 1.0$: pressure oscillations or pressure modes are eliminated but the accuracy of displacement and pressure solutions is not good.

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