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SOME CHARACTERIZATIONS OF SINGULAR COMPACTIFICATIONS

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ABSTRACT. Assume that X is locally compact and Hausdorff. Then, we show that $\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$ for any compactification αX of X if and only if for any 2-point compactification γX of X with $\gamma X - X = \{-\infty, +\infty\}$, there exists a clopen subset A of γX such that $-\infty \in A$ and $+\infty \notin A$. As a corollary, we obtain that if X is connected and locally connected, then $\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$ for any compactification αX of X if and only if X is 1-complemented.

Throughout this paper, all topological spaces concerned are assumed to be Hausdorff and the space X to be noncompact and locally compact.

Let Y be compact and let $f : X \to Y$ be continuous with $f(X)$ dense in Y. The subset $S(f)$ of Y defined by $\{p \in Y |$ for any neighborhood U of p, the closure of $f^{-1}(U)$ in Y is not compact $\}$ is called the singular set of f. And also, f is called singular([2],[3]) if $S(f) = Y$.

For a singular map $f : X \to Y$, the singular compactification of X induced by f, which is denoted by $X \cup_f S(f)$, is defined as follows([7],[9]);

On the set $X \cup S(f)$, basic neighborhoods of $p \in X$ are the same in X and $p \in S(f) = Y$ has basic neighborhoods of the form $V \cup \{f^{-1}(V) - F\}$, where V is a neighborhood of p and F is any compact subset in X.

Let $C^*(X)$ be the set of all continuous and bounded map from X to the real line R. For a compactification αX of X and f in $C^*(X)$, we denote f^α the extension of f to αX if exists. Let $C_\alpha(X)$ denote the set of f in $C^*(X)$ which have extension to αX, and $S^\alpha(S^*)$ denote the set of f in $C_\alpha(X)(C^*(X))$ which is singular. For $f \in C^*(X)$, since $Cl_R(f(X))$ is compact in R, we have that f is singular if and only if $S(f) = Cl_R(f(X))$. In this paper, we consider the following problem; 'Is $\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$ for any compactification αX of X ?'

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EXAMPLE 1. Consider the 2-point compactification γR of real line R. Then, $\gamma R \neq \sup\{R \cup_f S(f) | f \in S^\gamma\}$. For, suppose that $\gamma R = \sup\{R \cup_f S(f) | f \in S^\gamma\}$, then since the compactification which is strictly less than γR is the 1-point compactification, we have that $\gamma R = R \cup_f S(f)$ for some $f \in S^\gamma$. So, this implies that $\gamma R - R = S(f) = Cl_R(f(R))$, which is impossible since $\gamma R - R$ is a discrete space with 2 point and $Cl_R(f(R))$ is connected set in R.

This Example 1 gives an idea that the problem can be reduced to that of the 2-point compactifications as you will see in Theorem 5. We start with the following Theorem of Chandler and Faulkner.

THEOREM 2([6]). *Let αX be a compactification of X, and let $\mathcal{G}$ be a subcollection of S^α. Then, $\alpha X = \sup\{X \cup_f S(f) | f \in \mathcal{G}\}$ if and only if $\mathcal{G}^\alpha = \{f^\alpha | f \in S^\alpha\}$ separates points in $\alpha X - X$.*

LEMMA 3([6]). *For f in $C_\alpha(X)$, we have $f^\alpha(\alpha X - X) = S(f)$.*

Using Theorem 2 and Lemma 3, we have the following which is some modification of it.

LEMMA 4. *Let αX be a compactification of X.*

Then, $\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$ if and only if for any distinct points p and q in $\alpha X - X$, there exists a continuous map $h : \alpha X \to [0,1]$ such that $h(p) = 0, h(q) = 1$ and $h(X) \subset h(\alpha X - X)$.

PROOF. Suppose that $\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$ and let p and q be distinct points in $\alpha X - X$. Then, by Theorem 2, there exists an $f \in S^\alpha$ such that $f^\alpha(p) \neq f^\alpha(q)$. And also, we have that $f^\alpha(X) = f(X) \subset S(f) = f^\alpha(\alpha X - X)$ by Lemma 3 and the fact that f is singular. It is obvious that there exists a continuous map $g : R \to [0,1]$ such that $g(f^\alpha(p)) = 0$ and $g(f^\alpha(q)) = 1$ since R is normal and Hausdorff. Letting $h = g \circ f^\alpha$, we have the desired map. For the converse, let p and q be distinct points in $\alpha X - X$. Then, there exists a continuous map $h : \alpha X \to [0,1]$ such that $h(p) = 0, h(q) = 1$ and $h(X) \subset h(\alpha X - X)$. Let $f = h|_X$. Then $f \in C_\alpha$ with $f^\alpha = h$. Since $f^\alpha(\alpha X - X) \subset f^\alpha(\alpha X) = f^\alpha(Cl_{\alpha X} X) \subset Cl_R(f^\alpha(X)) \subset Cl_R(f^\alpha(\alpha X - X)) = f^\alpha(\alpha X - X)$ (last equality comes from that $\alpha X - X$ is compact since X is locally compact), we have that $S(f) = f^\alpha(\alpha X - X) = Cl_R(f^\alpha(X)) = Cl_R(f(X))$, that is,

f is singular such that f^α separates p and q. Hence, by Theorem 2, we have that $\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$. This completes the proof.

THEOREM 5. *The following statements are equivalent.*

(1) *For any compactification* αX, $\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$.

(2) *For any 2-point compactification* γX *of* X *with* $\gamma X - X = \{-\infty, +\infty\}$, *there exists a clopen(=closed and open) subset* A *of* γX *such that* $-\infty \in A$ *and* $+\infty \notin A$.

PROOF. (1)⇒(2). Let γX be 2-point compactification with $\gamma X - X = \{-\infty, +\infty\}$. Then, by Lemma 4, there exists a continuous map $h : \gamma X \to [0,1]$ such that $h(-\infty) = 0$, $h(+\infty) = 1$ and $h(X) \subset h(\gamma X - X) = \{0, 1\}$. Let $A = h^{-1}(\{0\})$. Then, A is a clopen subset of γX such that $-\infty \in A$ and $+\infty \notin A$.

(2)⇒(1). Let αX be any compactification of X, and let p and q be distinct points in $\alpha X - X$.

Case 1. p and q are in the same component, say U, of $\alpha X - X$:

Since αX is compact Hausdorff,there exists a continuous map h : $\alpha X \to [0,1]$ such that $h(p) = 0$ and $h(q) = 1$. Since $h(p) = 0, h(q) = 1$ and $h(U)$ is connected in $[0,1]$, we have that $h(U) = [0,1]$. So, $h(X) \subset [0,1] = h(U) \subset h(\alpha X - X)$.

Case 2. p and q are not in the same component of $\alpha X - X$:

Let U be the component of p in $\alpha X - X$, and let $\gamma X = X \cup \{-\infty, +\infty\}$ with the quotient topology determined by ϕ where $\phi : \alpha X \to \gamma X$ is defined by $\phi(x) = x$ if $x \in X, \phi(x) = -\infty$ if $x \in U$ and $\phi(x) = +\infty$ otherwise. Then, γX is a 2-point compactification of X. By hypothesis, there exists a clopen subset A of γX such that $-\infty \in A$ and $+\infty \notin A$. Define a continuous map $g : \gamma X \to [0,1]$ by $g(x) = 0$ if $x \in A$ and $g(x) = 1$ otherwise, and let $h = g \circ \phi$. Then, we have that $h(p) = 0, h(q) = 1$ and $h(X) = g \circ \phi(X) \subset \{0, 1\} = g(\{-\infty, +\infty\}) = g \circ \phi(\alpha X - X) = h(\alpha X - X)$.

In any case, we have that there exists a continuous map $h : \alpha X \to [0,1]$ such that $h(p) = 0, h(q) = 1$ and $h(X) \subset h(\alpha X - X)$. Therefore, by Lemma 4, we have that $\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$. This completes the proof.

COROLLARY 6. *If X has no 2-point compactifications, then $\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$ for any compactification αX of X.*

PROOF. Straightforward by Theorem 5.

COROLLARY 7. *Let X be a connected space. Then, the following statements are equivalent.*

(1) *X has no 2-point compactifications*
(2) *$\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$ for any compactification αX of X.*

PROOF. It is sufficient to prove that (2)⇒(1). If there exists 2-point compactification αX of X with $\alpha X - X = \{-\infty, +\infty\}$, then, by Theorem 5, there exists a continuous map $h : \alpha X \to [0,1]$ such that $h(-\infty) = 0, h(+\infty) = 1$ and $h(X) \subset h(\alpha X - X) = \{0,1\}$. So, we have that $h(\alpha X) = \{0,1\}$, which is impossible since $h(\alpha X)$ is connected.

A space X is said to be 1-complemented if each compact subset K is contained in some compact subset F with $X - F$ connected.

LEMMA 8([1]). *Let X be locally connected. Then, X has no 2-point compactifications if and only if X is 1-complemented.*

Combining Corollary 7 and Lemma 8, we have the following.

COROLLARY 9. *Let X be connected and locally connected. Then, the following statements are equivalent.*

(1) *X is 1-complemented.*
(2) *X has no 2-point compactifications*
(3) *$\alpha X = \sup\{X \cup_f S(f) | f \in S^\alpha\}$ for any compactification αX of X.*

As you see in the following Example 10, Corollary 9 doesn't hold if the local connectedness of X is deleted.

EXAMPLE 10. For $n = 0, 1, \cdots$, let $X_{2n} = \{(x,y) \mid y = 2n + 2kx, 0 \le x \le \frac{1}{k}, k = 1, 2, \cdots\}$, $X_{2n+1} = \{(x,y) \mid y = 2n+1-2kx, -\frac{1}{k} \le x \le 0, k = 1, 2, \cdots\}$ and $X = \{(0,y) \mid y \ge 0\}$. And also, let $Y = X \cup (\cup_{n=0}^{\infty} X_n)$. We call the subspace Y of the Eucidean space R^2 with usual topology '*Broom*'. It is trivial that Y is locally compact Hausdorff space which is connected, but not locally connected. Y is not 1-complemented. For, since $\{0\}$ is compact in Y, if there exists a compact subset F of Y such that $0 \in F$ and $Y - F$ is connected, then we have a contradiction that $F = Y$. To show that Y has no 2-point compactifications, by Lemma 8, it is sufficient to show that for any compact subset K of Y there exist a compact subset H and a connected subset C such that $K \subset H, H \cup C = Y$ and $K \cap C = \emptyset$. For any compact subset K, there exists an n such that $K \cap V_n = \emptyset$ where $V_n = Y \cap (-\infty, +\infty) \times (n, +\infty)$. For odd n, let $C = (Y \cap [0, +\infty) \times [n+1, n+2]) \cup V_{n+2}$ and for even n, let $C = (Y \cap (-\infty, 0] \times [n+1, n+2]) \cup V_{n+2}$. Letting $H = Y - V_{n+2}$, we have a compact subset H and a connected subset C with $K \subset H, H \cup C = Y$ and $K \cap C = \emptyset$.

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